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Large Time Behavior for Convection-Diffusion Equations in $\mathbb{R}^N$ with Periodic Coefficients

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We describe the large time behavior of solutions of the convection-diffusion equation

$$u_t - \operatorname{div}(a(x)\, \nabla u) = d \cdot \nabla(|u|^{q-1}\, u) \qquad \text{in} \quad (0, \infty) \times \mathbb{R}^N$$

where $d \in \mathbb{R}^N$ and $a = a(x)$ is a symmetric periodic matrix satisfying suitable ellipticity assumptions. We also assume that $a \in W^{1,\infty}(\mathbb{R}^N)$. First, we consider the linear problem ($d = 0$) and prove that the large time behavior of solutions is given by the fundamental solution of the diffusion equation with $a \equiv a^h$ where a^h is the homogenized matrix. In the nonlinear case, when $q = 1 + \frac{1}{N}$, we prove that the large time behavior of solutions with initial data in $L^1(\mathbb{R}^N)$ is given by a uniparametric family of self-similar solutions of the convection-diffusion equation with constant homogenized diffusion $a \equiv a^h$. When $q > 1 + \frac{1}{N}$, we prove that the large time behavior of solutions is given by the fundamental solution of the linear-diffusion equation with $a \equiv a^h$.

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1. INTRODUCTION

This paper is devoted to the study of the large time behavior of solutions of the following convection-diffusion equation

$$\begin{cases} u_t - \operatorname{div}(a(x)\,\nabla u) = d \cdot \nabla(|u|^{q-1}\,u) & \text{in} \quad (0, \infty) \times \mathbb{R}^N \\ u(0, x) = u_0(x) \end{cases} \tag{1}$$

with $q \geqslant 1 + \frac{1}{N}$, $N \geqslant 1$ and $d \in \mathbb{R}^N$ a constant vector. By $\cdot$ we denote the scalar product in $\mathbb{R}^N$.

We make the following assumptions on the coefficients a of the elliptic operator involved in (1) all along the paper:

- $a(x) = I + b(x)$ where I is the identity matrix and $b(x)$ is a symmetric matrix with Y-periodic coefficients, $Y = \prod_{j=0}^{N} (0, y_j^0) \subset \mathbb{R}^N$;
- Each $a_{ij} \in W^{1,\infty}(\mathbb{R}^N)$;
- There exist C_0, $C_1 > 0$ such that

$$C_1 |\zeta|^2 \geqslant \sum_{k,\, l=1}^{N} a_{k,\, l}(x)\, \zeta_k \zeta_l \geqslant C_0 \sum_{k,\, l=1}^{N} |\zeta_k|^2 = C_0 |\zeta|^2$$

for all $\zeta = (\zeta_1, ..., \zeta_N) \in \mathbb{R}^N$, with $0 < C_0 < C_1 < \infty$.

We recall that a function $f\colon \mathbb{R}^N \to \mathbb{R}^M$ is said to be Y-periodic if it is periodic of period y_j^0 in the direction x_j, for each $j = 1, ..., N$.

For every $u_0 \in L^1(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ with $r > \frac{Nq}{N+2}$, system (1) admits a unique solution $u \in BC([0, \infty); L^1(\mathbb{R}^N)) \cap L^q((0, T) \times \mathbb{R}^N)$, for every $T < \infty$, such that

$$\int_0^T \int_{\mathbb{R}^N} (a(x)\, \nabla u \cdot \nabla \psi - u \psi_t)\, dx\, dt$$

$$= \int_{\mathbb{R}^N} u_0 \psi(0)\, dx - \int_{\mathbb{R}^N} u(T)\, \psi(T)\, dx - \int_0^T \int_{\mathbb{R}^N} d \cdot \nabla \psi\, |u|^{q-1}\, u\, dx\, dt \tag{2}$$

for any ψ such that $\psi \in W^{1,\infty}([0, T]; BC^2(\mathbb{R}^N) \cap H^2(\mathbb{R}^N))$ for every $T > 0$ (see Proposition 1 of [9]). Moreover solutions satisfy the following regularity property

$$u \in C((0, \infty); W^{2,p}(\mathbb{R}^N)) \cap C^1((0, \infty); L^p(\mathbb{R}^N)) \tag{3}$$

for every $p \in (1, \infty)$.

On the other hand, integrating equation (1) over all $\mathbb{R}^N$ we deduce that the total mass of solutions is conserved for all time, i. e.,

$$\int_{\mathbb{R}^N} u(t, x)\, dx = \int_{\mathbb{R}^N} u_0(x)\, dx = M, \qquad \forall t > 0.$$

Furthermore the solution u of (1) satisfies the following L^p-estimates (see Proposition 1 of [9]):

$$\|u(t)\|_p \leqslant C_p \, \|u_0\|_1 \, t^{(-N/2)(1-1/p)} \qquad \forall t > 0. \tag{4}$$

This estimate suggests that the natural question to study is the large time behavior of $t^{N/2(1-1/p)}\, u(t, x)$ in $L^p(\mathbb{R}^N)$ for $1 \leqslant p \leqslant \infty$.

Before going into the nonlinear problem we analyze the linear equation with $d = 0$, i.e.,

$$u_t - \operatorname{div}(a(x)\, \nabla u) = 0 \qquad \text{in} \quad (0, \infty) \times \mathbb{R}^N. \tag{5}$$

Note that the convective term is not present in this case.

When $u_0 \in L^1(\mathbb{R}^N)$, we shall prove that the asymptotic behavior of solutions of problem (5) is given by the solutions of the following one

$$\begin{cases} u^h_t - \operatorname{div}(a^h\, \nabla u^h) = 0 \\ u^h(0, x) = M\delta, \end{cases} \tag{6}$$

where a^h is the homogenized matrix, i.e.,

$$a^h_{ik} = \left[a_{ik}(y) + a_{ij}(y) \frac{\partial w^k}{\partial y_j}(y) \right]^{\sim}, \tag{7}$$

with w^k the unique solution of

$$\begin{cases} w^k \in V = \{u \in H^1_{loc}(\mathbb{R}^N), \quad u \text{ is } Y\text{-periodic}\} \\ \displaystyle\int_Y a_{ij} \frac{\partial w^k}{\partial y_j} \frac{\partial v}{\partial y_i}\, dy = \int_Y \frac{\partial a_{ik}}{\partial y_i}\, v\, dy \qquad \forall v \in V; \\ \tilde{w}^k = 0, \end{cases} \tag{8}$$

and where $\sim$ denotes the average,

$$\tilde{f} = \frac{1}{|Y|} \int_Y f(y)\, dy,$$

$|Y|$ being the measure of Y.

We observe that the solution u^h of (6) has a self-similar structure, i.e., $u^h = M t^{-N/2} f(x/\sqrt{t})$ for a suitable f.

The proof of this result is done with scaling arguments and classical tecniques in homogenization theory. Notwithstanding, in the context of homogenization the initial data are usually considered as fixed. As we shall see, in our problem, we have to consider a sequence of initial data that weakly converges to a Dirac mass.

In order to understand the large time behavior of u solution of problem (1) and to see how homogenization theory enters in the problem we introduce the scaled functions

$$u_\lambda(t, x) = \lambda^N u(\lambda^2 t, \lambda x), \qquad \lambda > 0.$$

Those functions that remain invariant under this change of variable will be refered to as self-similar. More precisely, $u = u(t, x)$ is said to be self-similar if and only if $u_\lambda \equiv u$ for all $\lambda > 0$. Note that u is self-similar if and only if there exists a profile f such that $u(x, t) = t^{-N/2} f(x/\sqrt{t})$.

Let us observe that, if u solves (1), then u_λ solves

$$u_{\lambda, t} - \operatorname{div}(a(\lambda x)\, \nabla u_\lambda) = \lambda^{N(1-q)+1} d \cdot \nabla(|u_\lambda|^{q-1} u_\lambda), \tag{9}$$

where $a_\lambda(x) = I + b_\lambda(x)$, and $b_\lambda(x)$ is a matrix with Y-periodic coefficients $b_{\lambda, ij}(x) = b_{ij}(\lambda x)$ in the variable $\lambda x = y$. On the other hand, passing to the limit on u as $t \to \infty$ can be reduced to passing to the limit on u_λ at $t = 1$ as $\lambda \to \infty$. Thus we are interested on the limit of u_λ as $\lambda \to \infty$.

We have that the coefficient $b_{ij}(\lambda x)$ satisfies (see [19], Lemma 4.1, p. 57)

$$b_{ij}(\lambda x) \rightharpoonup \tilde{b}_{ij} \qquad \text{weakly in} \quad L^2(\Omega) \qquad \text{as} \quad \lambda \to \infty \tag{10}$$

for any bounded open set Ω of $\mathbb{R}^N$. However, this weak convergence in L^2 is not sufficient to pass to the limit in the equation (9) when $\lambda \to \infty$ and therefore we need to use the classical notion of H-convergence (see [3], [19]). For this notion of convergence it is well known that

$$a_\lambda \xrightarrow{\mathrm{H}} a^h \qquad \text{as} \quad \lambda \to \infty,$$

with a^h as in (7)–(8). Moreover, this convergence allows us to pass to the limit in the equation (9) to obtain (6). Thus, we expect the limit diffusion as $t \to \infty$ to be constant $a \equiv a^h$.

In the nonlinear case, we observe that:

- When $q > 1 + \frac{1}{N}$, the power $N(1-q)+1$ of λ in (9) is negative. Therefore, formally (this will be made precise below), the convection term should vanish as $\lambda \to \infty$.
- When $q = 1 + \frac{1}{N}$, we have that $\lambda^{N(1-q)+1} = 1$ for all $\lambda > 0$. Therefore, formally, all terms of the equation should remain unchanged when passing to the limit as $\lambda \to \infty$.

As a consequence of these remarks we should expect the three following results:

(a) When $d=0$, the large time behavior of solutions of the linear problem should be given by solutions of the homogenized problem (6). We refer to J. Ortega [17] for a complete asymptotic expasion using Bloch waves.

(b) When $q=1+\frac{1}{N}$ and $d\neq 0$ the asymptotic behavior of the solutions of (1) should be given by the self-similar solutions of the full equation with $a\equiv a^h$, i.e., the system should present a *self-similar behavior*.

(c) When $q>1+\frac{1}{N}$ and $d\neq 0$ the asymptotic behavior of the solutions of (1) should be given by the linear homogenized problem (6), i.e., the system should present a *weakly nonlinear behavior*.

This paper is devoted to prove these three results.

Note that we do not address the case $1<q<1+\frac{1}{N}$. When $a\equiv 1$ it is by now well known that solutions of (1) behave in a strongly nonlinear way as $t\to\infty$ since the diffusion vanishes asymptotically in the convective direction (see [11] and [12]). However, when a is periodic the techniques of these works do not allow to get sharp L^∞-decay rates. The description of the asymptotic behavior of (1) for non-constant periodic coefficients when $1<q<1+\frac{1}{N}$ remains open.

The rest of the paper is organized as follows. In Section 2, we recall some results on homogenization for elliptic equations. In Section 3 we study the asymptotic behavior as time tends to infinity of the solutions of the *linear problem.* In Section 4 we prove the result on the *weakly nonlinear asymptotic behavior* when $q>1+\frac{1}{N}$. Finally, in section 5 we prove the *self-similar asymptotic behavior* when $q=1+\frac{1}{N}$.

2. PRELIMINARIES ON HOMOGENIZATION OF ELLIPTIC EQUATIONS

In this section we present some classical results on homogenization of second order elliptic equations in divergence form.

DEFINITION 1. A sequence of coefficients a_λ is said to H-converge to a^h as $\lambda\to\infty$ if and only if for any $f\in H^{-1}(\mathbb{R}^N)$ the family of solutions $u_\lambda\in H^1(\mathbb{R}^N)$ of

$$-\operatorname{div}(a_\lambda \nabla u_\lambda)+u_\lambda=f \qquad \text{in} \quad \mathbb{R}^N$$

satisfies

$$\begin{aligned} u_\lambda &\rightharpoonup u \qquad \text{weakly in} \quad H^1(\mathbb{R}^N) \\ a_\lambda \nabla u_\lambda &\rightharpoonup a^h \nabla u \qquad \text{weakly in} \quad (L^2(\mathbb{R}^N))^N, \end{aligned}$$

where $u \in H^1(\mathbb{R}^N)$ is the solution of

$$-\operatorname{div}(a^h \nabla u) + u = f \qquad \text{in} \quad \mathbb{R}^N. \tag{11}$$

This convergence will be denoted by $a_\lambda \xrightarrow{\text{H}} a^h$.

The above definition was introduced by S. Spagnolo [21] (under the name of G-convergence in the case of symmetric matrices) and later on by L. Tartar [22] in the general case. An extensive literature on the topic is now available; see for instance the books of A. Bensoussan, J.-L. Lions, G. Papanicolaou [3] and E. Sanchez-Palencia [19].

Now, we recall the following classical result (see [19], p. 57). We also give a sketch of its proof for the sake of completeness.

PROPOSITION 1. *Let $a_\lambda(x) = a(\lambda x)$ with a Y-periodic and satisfying all the assumptions of the introduction. Then*

$$a^\lambda \xrightarrow{\text{H}} a^h \qquad \text{as} \quad \lambda \to \infty, \tag{12}$$

where a^h is the homogenized matrix whose coefficients are given by (7)–(8).

Proof. Let u_λ be the family of solutions of the problems

$$-\operatorname{div}(a_\lambda \nabla u_\lambda) + u_\lambda = f \qquad \text{in} \quad \mathbb{R}^N$$

with $f \in H^{-1}(\mathbb{R}^N)$ and with a_λ defined as above. Applying the Lax–Milgram Lemma we deduce that for $\lambda > 0$ fixed there exists a unique solution $u_\lambda \in H^1(\mathbb{R}^N)$ of

$$\int_{\mathbb{R}^N} a_\lambda \nabla u_\lambda \cdot \nabla \phi \, dx + \int_{\mathbb{R}^N} u_\lambda \phi \, dx = \int_{\mathbb{R}^N} f\phi \, dx \tag{13}$$

for all $\phi \in H^1(\mathbb{R}^N)$. Analogously, the solution u of (11) belongs to $H^1(\mathbb{R}^N)$ and verifies (13) with a^h and u instead of a_λ and u_λ.

Taking (1) into account and taking $\phi = u_\lambda$ in (13) we have

$$C_0 \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 \, dx + \int_{\mathbb{R}^N} |u_\lambda|^2 \, dx \leqslant \int_{\mathbb{R}^N} \nabla u_\lambda a_\lambda \nabla u_\lambda \, dx + \int_{\mathbb{R}^N} |u_\lambda|^2 \, dx$$

and

$$\int_{\mathbb{R}^N} f u_\lambda \, dx \leqslant \|f\|_{H^{-1}(\mathbb{R}^N)} \; \|u_\lambda\|_{H^1(\mathbb{R}^N)}.$$

We deduce

$$\|u_\lambda\|_{H^1(\mathbb{R}^N)} \leqslant C$$

with C independent of λ. Consequently, we can extract a subsequence $\lambda_i \to \infty$ such that

$$u_{\lambda_i} \rightharpoonup u^* \qquad \text{weakly in} \quad H^1(\mathbb{R}^N). \tag{14}$$

We have to see that $u^* = u$, where u is the solution of (11). From (14), the gradient of u_λ is bounded in $L^2(IR^N)$. Multiplying it by a_λ (bounded in L^∞) we obtain that

$$\|a_\lambda \, \nabla u_\lambda\|_{L^2(\mathbb{R}^N)} \leqslant C.$$

Using the notation $P_\lambda = a_\lambda \, \nabla u_\lambda$, we can extract a subsequence $\lambda_i \to \infty$ such that

$$P_{\lambda_i} \rightharpoonup P^* \qquad \text{weakly in} \quad L^2(\mathbb{R}^N). \tag{15}$$

By (13) we have that

$$\int_{\mathbb{R}^N} P_{\lambda_i} \cdot \nabla\phi \, dx + \int_{\mathbb{R}^N} u_{\lambda_i} \phi \, dx = \int_{\mathbb{R}^N} f\phi \, dx$$

for all $\phi \in H^1(\mathbb{R}^N)$, and passing to the limit as $\lambda_i \to \infty$:

$$\int_{\mathbb{R}^N} P^* \cdot \nabla\phi \, dx + \int_{\mathbb{R}^N} u^* \phi \, dx = \int_{\mathbb{R}^N} f\phi \, dx \quad \forall \phi \in H^1(\mathbb{R}^N). \tag{16}$$

Thus, it is sufficient to show that

$$P^* = a^h \, \nabla u^* \qquad \text{in} \quad \mathbb{R}^N. \tag{17}$$

Identity (17) can be rewritten as

$$P_i^*(x) = a_{ij}^h \frac{\partial u^*}{\partial x_j}(x) \qquad \text{in} \quad \mathbb{R}^N$$

for every $i = 1, ..., N$.

We fix any $k=1, ..., N$. If $w^k(y)$ is the function defined by (8), we set

$$w_\lambda(x) = x^k + \frac{1}{\lambda} w^k(\lambda x).$$

Therefore we have that

$$w_\lambda(x) \to x^k \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \lambda \to \infty \tag{18}$$

for any open bounded set Ω of $\mathbb{R}^N$. Moreover, w_λ satisfies the equation

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(\lambda x)\frac{\partial w_\lambda}{\partial x_j}(x)\right) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

Then, multiplying in this identity by a test function $\phi \in H_0^1(\Omega)$ and integrating we have

$$\int_\Omega a_{ij}(\lambda x)\frac{\partial w_\lambda}{\partial x_j}\frac{\partial \phi}{\partial x_i}\, dx = 0. \tag{19}$$

Let $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^N)$ be with support in Ω. We take $\phi = \varphi w_\lambda$ in (13) and $\phi = \varphi u_\lambda$ in (19). Substracting these two identities and taking into account that $a_{ij} = a_{ji}$ we have

$$\int_\Omega a_{ij}(\lambda x)\left[\frac{\partial u_\lambda}{\partial x_j}\frac{\partial \varphi}{\partial x_i} w_\lambda - \frac{\partial w_\lambda}{\partial x_i}\frac{\partial \varphi}{\partial x_j} u_\lambda\right] dx + \int_\Omega u_\lambda w_\lambda \varphi\, dx = \int_\Omega f w_\lambda \varphi\, dx. \tag{20}$$

Now we can pass to the limit as $\lambda \to \infty$ in (20) because each term is the product of a term which converges weakly in $L^2(\Omega)$ and another one that converges strongly in $L^2(\Omega)$ for any bounded domain Ω. Indeed:

- $P_{\lambda,i}(x) \equiv a_{ij}(\lambda x)\frac{\partial u_\lambda}{\partial x_j}(x)$ converges weakly to P^* in L^2 by (15);
- $\frac{\partial \varphi}{\partial x_i} w_\lambda$ converges to $\frac{\partial \varphi}{\partial x_i} x^k$ in $L^2(\Omega)$ strongly by (18) (for φ fixed);
- $a_{ij}(\lambda x)\frac{\partial w_\lambda}{\partial x_j}(x)$ is $\frac{Y}{\lambda}-$periodic and tends weakly in L^2 to its mean value

$$\left[a_{ij}(y)\left(\delta_{ik} + \frac{\partial w^k}{\partial y_i}(y)\right)\right]^{\sim} = a_{jk}^h;$$

- $\frac{\partial \varphi}{\partial x_j} u_\lambda$ converges to $\frac{\partial \varphi}{\partial x_j} u^*$ in $L^2(\Omega)$ strongly, since, applying the Rellich Theorem in (14), we have that $u_{\lambda_i} \to u^*$ in $L^2(\Omega)$ and φ is of compact support;

• Finally, by the same reason, φu_λ converges to φu^* in $L^2(\Omega)$ strongly.

We obtain

$$\int_\Omega (P_j^* x_k - a_{jk}^h u^*) \frac{\partial \varphi}{\partial x_j} dx + \int_\Omega u^* x_k \varphi \, dx = \int_\Omega f x_k \varphi \, dx. \tag{21}$$

Moreover, applying (16) with $\phi = \varphi x_k$ we have

$$\int_\Omega f x_k \varphi \, dx = \int_\Omega P_j^* \frac{\partial(\varphi x_k)}{\partial x_j} dx + \int_\Omega u^* x_k \varphi \, dx. \tag{22}$$

From (21) and (22) we obtain that

$$\int_\Omega (P_j^* x_k - a_{jk}^h u^*) \frac{\partial \varphi}{\partial x_j} dx = \int_\Omega P_j^* \frac{\partial(\varphi x_k)}{\partial x_j} dx$$

for all $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^N)$. This means that, in the sense of distributions on $\mathbb{R}^N$, we have

$$-\frac{\partial}{\partial x_j} (P_j^* x_k - a_{jk}^h u^*) = -\frac{\partial P_j^*}{\partial x_j} x_k \Leftrightarrow P_k^* = a_{jk}^h \frac{\partial u^*}{\partial x_j}$$

which is equivalent to (17). The proof of Proposition 1 is now completed. ▮

Remark 1. Let us consider a bounded family f_λ in $H^{-1}(\mathbb{R}^N)$. Let u_λ be the family of solutions of

$$-\operatorname{div}(a_\lambda \nabla u_\lambda) + u_\lambda = f_\lambda \qquad \text{in} \quad \mathbb{R}^N.$$

Let us assume that

$$f_\lambda \to f \qquad \text{strongly in} \quad H^{-1}(\Omega)$$

for all open bounded set Ω of $\mathbb{R}^N$. Then, reproducing the proof of Proposition 1, we deduce that

$$u_\lambda \rightharpoonup u \qquad \text{weakly in} \quad H^1(\mathbb{R}^N),$$

where u is the solution of

$$-\operatorname{div}(a^h \nabla u) + u = f \qquad \text{in} \quad \mathbb{R}^N.$$

3. THE LINEAR CASE

In this section we prove that, in a first approximation, solutions of (5) behave like the fundamental solution of the homogenized problem. More precisely, the following result is proved:

THEOREM 1. *Let $u_0 \in L^1(\mathbb{R}^N)$ be such that $\int_{\mathbb{R}^N} u_0 = M$. Then, the unique solution $u = u(t, x)$ of (5) verifies*

$$t^{N/2(1-1/p)} \|u(t) - u^h(t)\|_p \to 0 \qquad as \quad t \to \infty \tag{23}$$

for all $p \in [1, \infty)$, where $u^h(t)$ is the unique solution of (6), which is of self-similar form. Moreover, for $N=1$ and $N=2$, (23) also holds for $p=\infty$.

Remark 2. When $N \leqslant 2$ the Theorem 1 is also satisfied if we assume less regularity on the coefficients, i.e., the assumption $a_{ij} \in L^\infty(\mathbb{R}^N)$ suffices instead of $a_{ij} \in W^{1,p}(\mathbb{R}^N)$. Indeed, going back to (59) we observe that, when $N \leqslant 2$, the following fact can be used:

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} u_\lambda(\delta, x)\, \psi_2(x)\, dx - \int_{\mathbb{R}^N} u_\lambda(0, x)\, \psi_2(x)\, dx \right| \\
&\quad \leqslant \left| \int_0^\delta \int_{\mathbb{R}^N} \partial_i u_\lambda(t, x)\, a_{i,j}(\lambda x)\, \partial_j \psi_2(x)\, dx\, dt \right| \\
&\quad \leqslant \|a(\lambda x)\|_\infty \, \|\nabla \psi_2\|_2 \int_0^\delta \|\nabla u_\lambda(t)\|_2\, dt.
\end{aligned}$$

Using (24) we observe that the last integral tends to 0 as $\delta \to 0$ uniformly in $\lambda > \lambda_0 > 0$ when $N \leqslant 2$.

In order to prove Theorem 1 we need the following Proposition:

PROPOSITION 2. *The following estimates hold:*

(a) *If $N \geqslant 1$ there is a constant $C > 0$ such that*

$$\|\nabla u(t)\|_2 \leqslant C \, \|u_0\|_1 \; Ct^{-N/4-1/2} \qquad \forall t > 0; \tag{24}$$

(b) *If $N \geqslant 2$ there exists $r^* > 2$ such that for any $2 < r < r^*$ there is a constant $C_r > 0$ such that*

$$\|\nabla u(t)\|_r \leqslant C_r \, \|u_0\|_1 \; Ct^{(-N/2)(1-1/r)-1/2} \qquad \forall t > 0, \tag{26}$$

for any solution of (5) with initial data $u_0(x) \in L^1(\mathbb{R}^N)$.

Proof of Proposition 2.

(a) Let $T(t)$ be the semigroup associated to the equation (5). Since the coefficients of the matrix $a(x)$ are in $L^\infty(\mathbb{R}^N)$ and in view of the ellipticity of a (see (1)), we deduce that (see Theorem 2.3.33 of [5])

$$\|\nabla u(2t)\|_2 \leqslant C \|u(t)\|_2 \, t^{-1/2}.$$

Combining this inequality with (4) for $p=2$ we deduce (24).

(b) In order to obtain (25) we shall need the following lemma:

LEMMA 1. *Assume that $N \geqslant 2$. Let $f \in L^q(\mathbb{R}^N)$ with $1 < q < N$. Then there is some $g \in (L^{Nq/(N-q)}(\mathbb{R}^N))^N$ such that*

$$f = \operatorname{div}(g). \tag{26}$$

Moreover, there exists a constant $C(N, q) > 0$ such that

$$\|g\|_{Nq/(N-q)} \leqslant C(N, q) \, \|f\|_q, \quad \forall f \in L^q(\mathbb{R}^N).$$

Assuming for the moment that this lemma holds, let us prove Proposition 2. Taking into account that $T(t)$ is an analitic semigroup in $L^p(\mathbb{R}^N)$ for every $p \in (1, \infty)$ (see Lemma 1.1 of [7]), we deduce that for every $p \in (1, \infty)$, there exists some $C = C(N, p, a)$, such that (see Theorem 2.5.4 of [18])

$$\|\operatorname{div}(a(x) \nabla u(t+\tau))\|_p \leqslant Ct^{-1} \, \|u(\tau)\|_p.$$

Therefore taking $t = \tau$ and taking into account (4) we obtain that

$$\|\operatorname{div}(a(x) \nabla u)\|_p \leqslant Ct^{(-N/2)(1-1/p)-1} \, \|u_0\|_1 \tag{27}$$

for every $p \in (1, \infty)$ and $C > 0$.

Let $t > 0$ be fixed and $f = \operatorname{div}(a(x) \nabla u)$. We apply Lemma 1 with $q \in (1, N)$ of the form $q = rN/(N+r)$. Taking into account that $Nq/(N-q) = r$ for this choice of q, in view of Lemma 1 we deduce the existence of $g \in (L^r(\mathbb{R}^N))^N$ such that

$$\operatorname{div}(a(x) \nabla u) = \operatorname{div}(g), \tag{28}$$

and

$$\|g\|_r \leqslant C \, \|\operatorname{div}(a(x) \nabla u)\|_{Nr/(N+r)} \tag{29}$$

with $C > 0$ (which depends in particular on r but not on N and t.)

Combining (27) and (29) we deduce that

$$\|g\|_r \leqslant Ct^{(-N/2)(1-1/r)-1/2}\,\|u_0\|_1. \tag{30}$$

On the other hand, in view of Theorem 1 of [16], it follows that there exists $r^*>2$ such that for any $2<r<r^*$ there is $C>0$ such that

$$\|\nabla u(t)\|_r \leqslant C\,\|g\|_r \tag{31}$$

for any pair u and g satisfying (28).

Combining (30)–(31), we deduce that for $2<r<r^*$ there exists $C=C(r)$ such that

$$\|\nabla u(t)\|_r \leqslant Ct^{(-N/2)(1-1/r)-1/2}\,\|u_0\|_1.$$

The proof of Proposition 2 will be concluded once Lemma 1 is proved.

Proof of Lemma 1. By Sobolev's inequality we know that if $p<N$, there is some $C=C(p,N)>0$ such that:

$$\|g\|_{Np/(N-p)} \leqslant C\,\|\nabla g\|_p, \qquad \forall g\in W^{1,p}(\mathbb{R}^N).$$

We prove (26) by duality. In fact we search g such that $g=\nabla v$, with v solution of

$$\Delta v = f. \tag{32}$$

Now we analyze the regularity of the solution of (32) when $f\in L^q(\mathbb{R}^N)$. For this we consider the adjoint problem:

$$\Delta\varphi = \operatorname{div}(h). \tag{33}$$

By Calderón-Zygmund's Theorem (see Corollary 9.10 of [15]) we know that $h\in(L^r(\mathbb{R}^N))^N$ implies $\nabla\varphi\in(L^r(\mathbb{R}^N))^N$ with continuity for every $1<r<\infty$. By Sobolev's embedding we have $\varphi\in L^{Nr/(N-r)}(\mathbb{R}^N)$ if $r<N$. From (32) and (33) we obtain that

$$\int f\varphi = -\int \nabla v\cdot h. \tag{34}$$

Taking r such that $\frac{Nr}{N-r}=\frac{q}{q-1}$, by Sobolev's embedding (since $r=\frac{Nq}{N(q-1)+q}<N$), we deduce that

$$\|\varphi\|_{q/(q-1)} \leqslant C\,\|\nabla\varphi\|_r$$

and therefore

$$\left|\int \nabla v \cdot h\right| = \left|\int f\varphi\right| \leqslant \|f\|_q \|\varphi\|_{q/(q-1)} \leqslant C \|f\|_q \|\nabla\varphi\|_r \leqslant C \|f\|_q \|h\|_r. \tag{35}$$

By (34)–(35) we conclude that $\nabla v \in (L^{r/(r-1)}(\mathbb{R}^N))^N$ and $\frac{r}{r-1} = \frac{Nq}{N-q}$. This concludes the proof of Lemma 1.

The proof of Proposition 2 is now completed. ∎

Proof of Theorem 1. We proceed in several steps.

Step 1. First, we observe that if u is the solution of (5), then $u_\lambda(t, x) = \lambda^N u(\lambda^2 t, \lambda x)$ is the solution of

$$\begin{cases} u_{\lambda, t} - \operatorname{div}(a(\lambda x)\, \nabla u_\lambda) = 0 & \text{in} \quad (0, \infty) \times \mathbb{R}^N \\ u_\lambda(0, x) = u_{\lambda, 0}(x) = \lambda^N u_0(\lambda x). \end{cases} \tag{36}$$

On the other hand, u^h is self-similar since the coefficients a^h are constant. Thus, proving (23) is equivalent to prove that

$$u_\lambda(1) \to u^h(1) \qquad \text{in} \quad L^p(\mathbb{R}^N) \qquad \text{as} \quad \lambda \to \infty$$

for all $p \in [1, \infty)$.

The proof of the latter consists in reducing the parabolic problem to an elliptic problem. Such a reduction can be performed through multiplication of the equation by a time-dependent test function $\varphi \in W^{1,\infty}[0, T]$ and defining, for any w in $C((0, \infty); L^2(\mathbb{R}^N))$, the following two auxiliary functions

$$\hat{w}(x) = \int_\delta^T w(t, x)\, \varphi(t)\, dt \tag{37}$$

$$\check{w}(x) = -\int_\delta^T w(t, x) \frac{\partial \varphi}{\partial t}(t)\, dt. \tag{38}$$

with $0 < \delta < T$.

Step 2: *Uniform estimates for the scaled solutions.* As a consequence of (4) we get:

$$\|u_\lambda(t)\|_p \leqslant C_p \|u_0\|_1\, t^{(-N/2)(1-1/p)}, \qquad \forall t > 0 \tag{39}$$

for all $p \in [1, \infty]$ with $C_p > 0$ independent of λ.

Combining (24) with (39) for $p = 2$ we deduce

$$\|\nabla u_\lambda(t)\|_2 \leqslant C \|u_0\|_1\, t^{-N/4-1/2}, \qquad \forall t > 0 \tag{40}$$

with $C>0$ independent of λ. We also have the following uniform bound as $|x| \to \infty$:

PROPOSITION 3. *Let u_λ be solution of problem* (5). *Then*

$$\|u_\lambda(t, \cdot)\|_{L^1(|x|>R)} \to 0 \qquad as \quad R \to \infty \tag{41}$$

uniformly in $\lambda \geqslant 1$ and $t \in [0, T]$, for any $T>0$ fixed.

Proof of Proposition 3. We have

$$\int_{|x|>R} |u_\lambda(t, x)| \, dx = \int_{|x|>\lambda R} |u(\lambda^2 t, x)| \, dx.$$

On the other hand, thanks to Theorem 1 of [1] we have that the fundamental solution of (5) can bounded above by the heat kernel

$$0 \leqslant S(t, \tau, x, \xi) \leqslant KG(c(t-\tau), x-\xi)$$

for some positive constants K and c, where $G(t, x) = (4\pi t)^{-N/2} \exp[-\frac{|x|^2}{4t}]$.

Therefore, taking into account that

$$u(t, x) = \int_{\mathbb{R}^N} S(t, 0, x, \xi) \, u_0(\xi) \, d\xi$$

we deduce that

$$\begin{aligned}
&\int_{|x|>\lambda R} |u(\lambda^2 t, x)| \, dx \\
&\qquad \leqslant \int_{|x|>\lambda R} \left[\int_{\mathbb{R}^N} |S(\lambda^2 t, 0, x, \xi)| \, |u_0(\xi)| \, d\xi \right] dx \\
&\qquad \leqslant C \int_{|x|>\lambda R} G(c\lambda^2 t) * |u_0| \, dx = C \, \|G(ct) * |u_{0,\lambda}|\|_{L^1(|x|>R)}.
\end{aligned}$$

Thus to prove (41) it is enough to see that

$$\|G(ct) * |u_{0,\lambda}|\|_{L^1(|x|>R)} \to 0$$

as $R \to \infty$, uniformly in $\lambda \geqslant \lambda_0 > 0$ and $t \in [0, T]$.

We consider $v_\lambda(t, x) = G(t) * |u_{0,\lambda}(x)|$ solution of

$$\begin{cases} v_{\lambda,t} - \Delta v_\lambda = 0 \\ v_\lambda(0, x) = |u_{0,\lambda}(x)| \end{cases}$$

and we define a radial function $\phi \in BC^2(\mathbb{R}^N)$ such that

$$\phi(r) = \begin{cases} 0, & \text{if} \quad 0 < r < 1 \\ 1, & \text{if} \quad r > 2 \end{cases}$$

and $0 \leqslant \phi \leqslant 1$ when $1 \leqslant r \leqslant 2$.

We set $v_{\lambda, R} = v_\lambda \phi_R$, where $\phi_R(x) = \phi(\frac{x}{R})$ for every $R > 0$. Then, taking into account the equation that v_λ satisfies, we obtain that $v_{\lambda, R}$ satisfies

$$\begin{cases} v_{\lambda, R, t} - \Delta v_{\lambda, R} = -\dfrac{2}{R} \nabla v_\lambda \cdot (\nabla \phi)_R - \dfrac{1}{R^2} v_\lambda (\Delta \phi)_R \\ v_{\lambda, R}(0, x) = |u_{0, \lambda}(x)| \, \phi_R(x), \end{cases}$$

where $(\nabla \phi)_R(x) = \nabla \phi(\frac{x}{R})$ and $(\Delta \phi)_R\,(x) = \Delta \phi(\frac{x}{R})$. Thus, $v_{\lambda, R}$ satisfies the following integral equation:

$$v_{\lambda, R}(t) = G(t) * |u_{0, \lambda}| \, \phi_R - \frac{2}{R} \int_0^t G(t-s) * \nabla v_\lambda \cdot (\nabla \phi)_R$$
$$- \frac{1}{R^2} \int_0^t G(t-s) * v_\lambda (\Delta \phi)_R.$$

Taking L^1-norms we obtain

$$\|v_{\lambda, R}(t)\|_{L^1(\mathbb{R}^N)} \leqslant \|G(t) * |u_{0, \lambda}| \, \phi_R\|_{L^1(\mathbb{R}^N)}$$
$$+ C \frac{2}{R} \int_0^t \|G(t-s) * \nabla v_\lambda \cdot (\nabla \phi)_R\|_{L^1(\mathbb{R}^N)}$$
$$+ \frac{1}{R^2} \int_0^t \|G(t-s) * v_\lambda (\Delta \phi)_R\|_{L^1(\mathbb{R}^N)}.$$

As $\|G(t)\|_1 = 1$ we have

$$\|v_{\lambda, R}(t)\|_{L^1(\mathbb{R}^N)} \leqslant C \, \||u_{0, \lambda}| \, \phi_R\|_{L^1(\mathbb{R}^N)} + \frac{C}{R} \int_0^t \|\nabla v_\lambda\|_{L^1(\mathbb{R}^N)} + \frac{C}{R^2} \int_0^t \|v_\lambda\|_{L^1(\mathbb{R}^N)}.$$

Therefore, we have

$$\|v_{\lambda, R}(t)\|_{L^1(\mathbb{R}^N)} \leqslant C \, \||u_{0, \lambda}| \, \phi_R\|_{L^1(\mathbb{R}^N)} + \frac{Ct^{1/2}}{R} + \frac{Ct}{R^2},$$

since

$$\|v_\lambda(t)\|_{L^1(\mathbb{R}^N)} = \|v_\lambda(0)\|_{L^1(\mathbb{R}^N)} = \|u_\lambda(0)\|_{L^1(\mathbb{R}^N)} = \|u(0)\|_{L^1(\mathbb{R}^N)}$$

and

$$\|\nabla v_\lambda\|_{L^1(\mathbb{R}^N)} \leqslant Ct^{-1/2} \|v_\lambda(0)\|_{L^1(\mathbb{R}^N)} \leqslant C \|u(0)\|_{L^1(\mathbb{R}^N)} t^{-1/2}.$$

On the other hand, taking the definition of ϕ_R into account we have

$$\| |u_{0,\lambda}| \phi_R\|_{L^1(\mathbb{R}^N)} \leqslant \|u_{0,\lambda}\|_{L^1(|x|>R)} = \|u_0\|_{L^1(|x|>\lambda R)} \to 0 \qquad \text{as} \quad R \to \infty,$$

uniformly for $\lambda \geqslant \lambda_0 > 0$. Therefore we deduce

$$\|v_{\lambda, R}(t)\|_{L^1(\mathbb{R}^N)} \leqslant C \|u_0\|_{L^1(|x|>\lambda R)} + \frac{Ct^{1/2}}{R} + \frac{Ct}{R^2}.$$

Finally, as $u_0 \in L^1(\mathbb{R}^N)$ and

$$\|v_\lambda(t)\|_{L^1(|x|>2R)} = \|v_{\lambda, R}(t)\|_{L^1(|x|>2R)} \leqslant \|v_{\lambda, R}(t)\|_{L^1(\mathbb{R}^N)}$$

we obtain

$$\|v_\lambda(t)\|_{L^1(|x|>2R)} \to 0 \qquad \text{as} \quad R \to \infty$$

uniformly in $\lambda \geqslant \lambda_0 > 0$ and $t \in [0, T]$ for any T fixed.

The proof of Proposition 3 is now completed. ∎

By (39) and (40) we can estract a subsequence u_{λ_i} such that

$$u_{\lambda_i} \rightharpoonup u^* \qquad \text{weakly in} \quad L^2((\tau, \infty); H^1(\mathbb{R}^N)) \qquad \text{as} \quad \lambda_i \to \infty \tag{42}$$

for every $\tau > 0$. We deduce that, for any $0 < \delta < T$, and $\varphi \in W^{1,\infty}[0, T]$

$$\hat{u}_{\lambda_i} \rightharpoonup \hat{u}^* \qquad \text{weakly in} \quad H^1(\mathbb{R}^N) \qquad \text{as} \quad \lambda_i \to \infty \tag{43}$$

$$\breve{u}_{\lambda_i} \rightharpoonup \breve{u}^* \qquad \text{weakly in} \quad H^1(\mathbb{R}^N) \qquad \text{as} \quad \lambda_i \to \infty. \tag{44}$$

Step 3: *Passage to the limit in the variational formulation.* Let $\psi \in W^{1,\infty}([0, T]; BC^2(\mathbb{R}^N) \cap H^2(IR^N))$ be a test function. In this step, we are going to pass to the limit as $\lambda \to \infty$ in the variational formulation of system (36):

$$\int_0^T \int_{\mathbb{R}^N} (a(\lambda x) \nabla u_\lambda \cdot \nabla \psi - u_\lambda \psi_t) \, dx \, dt = \int_{\mathbb{R}^N} u_{0,\lambda} \psi(0) \, dx - \int_{\mathbb{R}^N} u_\lambda(T) \psi(T) \, dx.$$

We fix $0<\delta<T$. It also follows that

$$\int_\delta^T \int_{\mathbb{R}^N} (a(\lambda x)\, \nabla u_\lambda \cdot \nabla \psi - u_\lambda \psi_t)\, dx\, dt$$

$$= \int_{\mathbb{R}^N} u_\lambda(\delta)\, \psi(\delta)\, dx - \int_{\mathbb{R}^N} u_\lambda(T)\, \psi(T)\, dx. \tag{45}$$

We are going to pass to the limit twice in (45). First as $\lambda \to \infty$ and then as $\delta \to 0$.

To pass to the limit in (45) it is enough to consider test functions of the form $\psi(t,x) = \psi_1(t)\, \psi_2(x)$ with $\psi_1 \in W^{1,\infty}[0,T]$ and $\psi_2 \in BC^2(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$. Once this is done, using a classical density argument we can easily extend the limit process to all test functions $\psi \in W^{1,\infty}([0,T]; BC^2(\mathbb{R}^N) \cap H^2(\mathbb{R}^N))$ (see page 56 of [6]).

Let $\psi(t,x) = \psi_1(t)\, \psi_2(x)$. We can write (45) as

$$\int_{\mathbb{R}^N} [a(\lambda x)\, \nabla \hat{u}_\lambda \cdot \nabla \psi_2 + \breve{u}_\lambda \psi_2]\, dx$$

$$= \psi_1(\delta) \int_{\mathbb{R}^N} u_\lambda(\delta)\, \psi_2\, dx - \psi_1(T) \int_{\mathbb{R}^N} u_\lambda(T)\, \psi_2\, dx,$$

with $\hat{u}_\lambda$ and $\breve{u}_\lambda$ defined as in (37) and (38) and $\varphi = \psi_1$.

Therefore, we have that $\hat{u}_\lambda$ satisfies the following elliptic problem:

$$-\operatorname{div}(a_\lambda\, \nabla \hat{u}_\lambda) + \hat{u}_\lambda = f_\lambda \qquad \text{in} \quad \mathbb{R}^N,$$

where

$$f_\lambda = -\breve{u}_\lambda + \hat{u}_\lambda - u_\lambda(T)\, \psi_1(T) + u_\lambda(\delta)\, \psi_1(\delta).$$

By Proposition 1 we have $a^\lambda \xrightarrow{\mathrm{H}} a^h$ as $\lambda \to \infty$. Therefore if we prove that f_λ is bounded in $H^{-1}(\mathbb{R}^N)$ and that

$$f_\lambda \to f \qquad \text{strongly in} \quad H^{-1}(\Omega), \tag{46}$$

with $f = -\breve{u}^* + \hat{u}^* - u^*(T)\, \psi_1(T) + u^*(\delta)\, \psi_1(\delta)$ for every bounded domain Ω of $\mathbb{R}^N$, by the Remark 1 we deduce that u^* satisfies

$$-\operatorname{div}(a^h\, \nabla \hat{u}^*) = -\breve{u}^* - u^*(T)\, \psi_1(T) + u^*(\delta)\, \psi_1(\delta) \qquad \text{in} \quad \mathbb{R}^N,$$

for any ψ_1.

By (39) we deduce that f_λ is bounded in $L^2(\mathbb{R}^N)$ and therefore in $H^{-1}(\mathbb{R}^N)$. Thanks to (43)–(44) and that $H^1(\Omega)$ is compactly embedded in

$H^{-1}(\Omega)$, we may pass to the limit in the first two terms of f_λ, i.e., we have that

$$-\breve{u}_\lambda + \hat{u}_\lambda \to -\breve{u}^* + \hat{u}^* \qquad \text{strongly in} \quad H^{-1}(\Omega) \qquad \text{as} \quad \lambda \to \infty.$$

We claim that

$$u_\lambda(T)\,\psi_1(T) \to u^*(T)\,\psi_1(T) \qquad \text{strongly in} \quad H^{-1}(\Omega), \tag{47}$$

and that

$$u_\lambda(\delta)\,\psi_1(\delta) \to u^*(\delta)\,\psi_1(\delta) \qquad \text{strongly in} \quad H^{-1}(\Omega), \tag{48}$$

as $\lambda \to \infty$. To prove (47), taking into account that $L^2(\Omega)$ is compactly embedded in $H^{-1}(\ \Omega)$, it is enough to prove that

$$u_\lambda(T) \to u^*(T) \qquad \text{weakly in} \quad L^2(B_R) \qquad \text{as} \quad \lambda \to \infty \tag{49}$$

for every $R > 0$, where B_R is the ball of radius R.

In fact, thanks to (39) and (40) we have

(a) $\{u_\lambda\}$ is uniformly bounded in $L^\infty_{loc}((0, \infty); H^1_{loc}(\mathbb{R}^N))$.

Using (40), the fact that $a(\lambda x) \in L^\infty(\mathbb{R}^N)$ and equation (9) we deduce that

(b) $\{\partial_t u_\lambda\}$ is uniformly bounded in $L^2_{loc}((0, \infty); H^{-1}(\Omega))$ for every bounded domain Ω of $\mathbb{R}^N$.

Taking into account that $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, combining (a) and (b) and applying classical compactness results (cf. [20], Corollary 4, p. 85) we deduce that

$$\{u_\lambda\} \text{ is relatively compact in } C([t_1, t_2]; L^2(\Omega)) \text{ for every } 0 < t_1 < t_2 < \infty. \tag{50}$$

Therefore, we obtain that

$$u_\lambda \to u^* \qquad \text{in} \quad C([t_1, t_2]; L^2(\Omega)). \tag{51}$$

Obviously (49) is a consequence of (51). The same argument shows that (48) holds.

Therefore we have that u^* satifies:

$$\int_{\mathbb{R}^N} [a^h\,\nabla\hat{u}^* \cdot \nabla\psi_2 + \breve{u}^*\psi_2]\,dx = \int_{\mathbb{R}^N} u^*(\delta)\,\psi_2\psi_1(\delta)\,dx - \int_{\mathbb{R}^N} u^*(T)\,\psi_2\psi_1(T)\,dx. \tag{52}$$

Now, if we pass to the limit as $\delta \to 0$ in the left hand side of (52) we see that

$$\int_\delta^T \int_{\mathbb{R}^N} (a^h \nabla u^* \cdot \nabla \psi_2 - u^* \psi_{2,t})\, dx\, dt \to \int_0^T \int_{\mathbb{R}^N} (a^h \nabla u^* \cdot \nabla \psi_2 - u^* \psi_{2,t})\, dx\, dt$$

since $a^h \nabla u^* \cdot \nabla \psi_2 \in L^1((0,T) \times \mathbb{R}^N)$ and $u^* \psi_{2,t} \in L^1((0,T) \times \mathbb{R}^N)$. Finally, we are going to see that

$$\lim_{\delta \to 0^+} \int_{\mathbb{R}^N} u^*(\delta, x)\, \psi_2(x)\, dx = M\psi_2(0) \tag{53}$$

for all $\psi_2 \in BC^2(\mathbb{R}^N)$. For this we first prove that

$$\int_{\mathbb{R}^N} u^*(\delta, x)\, \psi_2(x)\, dx = \lim_{\lambda \to \infty} \int_{\mathbb{R}^N} u_\lambda(\delta, x)\, \psi_2(x)\, dx. \tag{54}$$

In fact, taking (51) into account we have

$$u_\lambda(\delta) \to u^*(\delta) \qquad \text{strongly in} \quad L^2(B_R) \qquad \text{as} \qquad \lambda \to \infty \tag{55}$$

for every $R > 0$, where B_R is the ball of radius R. Therefore we deduce that, for every $R > 0$,

$$\int_{B_R} u_\lambda(\delta, x)\, \psi_2(x) \to \int_{B_R} u^*(\delta, x)\, \psi_2(x) \qquad \text{as} \qquad \lambda \to \infty.$$

On the other hand, using Fatou's Lemma and taking (41) into account we deduce that $\int_{B_R^c} u^*(\delta, x)\, \psi_2(x)$ and $\int_{B_R^c} u_\lambda(\delta, x)\, \psi_2(x)$ are uniformly small when R is large enough. This completes the proof of (54).

In view of (54), (53) is equivalent to

$$\lim_{\delta \to 0^+} \lim_{\lambda \to \infty} \int_{\mathbb{R}^N} u_\lambda(\delta, x)\, \psi_2(x)\, dx = M\psi_2(0) \tag{56}$$

for all $\psi_2 \in BC^2(\mathbb{R}^N)$.

Multiplying equation (36) by a test function $\psi_2 \in C_c^\infty(\mathbb{R}^N)$ and integrating in $(0, \delta) \times \mathbb{R}^N$ we have:

$$\int_{\mathbb{R}^N} u_\lambda(\delta, x)\, \psi_2(x)\, dx - \int_{\mathbb{R}^N} u_\lambda(0, x)\, \psi_2(x)\, dx$$

$$= \int_0^\delta \int_{\mathbb{R}^N} u_{\lambda, t} \psi_2\, dx\, dt$$

$$= \int_0^\delta \int_{\mathbb{R}^N} \nabla u_\lambda a(\lambda x)\, \nabla \psi_2\, dx\, dt$$

$$= -\int_0^\delta \int_{\mathbb{R}^N} u_\lambda \operatorname{div}(a(\lambda x)\, \nabla \psi_2)\, dx\, dt. \tag{57}$$

Taking into account that, with the convection of summation of the repeated indexes,

$$\operatorname{div}(a(\lambda x)\, \nabla \psi_2) = a_{ij}(\lambda x)\, \partial^2_{ij} \psi_2 + \lambda \partial_i a_{ij}(\lambda x)\, \partial_j \psi_2, \tag{58}$$

we have

$$\left| \int_{\mathbb{R}^N} u_\lambda(\delta, x)\, \psi_2(x)\, dx - \int_{\mathbb{R}^N} u_\lambda(0, x)\, \psi_2(x)\, dx \right|$$

$$\leqslant \left| \int_0^\delta \int_{\mathbb{R}^N} u_\lambda(t, x)\, a_{ij}(\lambda x)\, \partial^2_{ij} \psi_2(x)\, dx\, dt \right|$$

$$+ \left| \int_0^\delta \int_{\mathbb{R}^N} u_\lambda(t, x)\, \lambda\, \partial_i a_{ij}(\lambda x)\, \partial_j \psi_2(x)\, dx\, ds \right|$$

$$\leqslant C\, \|a_{ij}(\lambda x)\, \partial^2_{ij} \psi_2\|_\infty\, \delta + \|\nabla \psi_2\|_\infty\, \|\lambda \operatorname{div}(a(\lambda x))\|_p \int_0^\delta \|u_\lambda(t)\|_{p'}\, dt \tag{59}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$ where $\operatorname{div} a$ denotes the vector $\operatorname{div} a = (\operatorname{div} a_1, \operatorname{div} a_2, ..., \operatorname{div} a_N)$. If we take $\frac{N}{2} < p \leqslant N$, we obtain that

$$\|\lambda \operatorname{div}(a(\lambda x))\|_p = \lambda^{1 - N/p}\, \|\operatorname{div} a\|_p \tag{60}$$

with $1 - \frac{N}{p} \leqslant 0$. On the other hand, in view of (39), we have

$$\|u_\lambda(t)\|_{p'} \leqslant C t^{(-N/2)(1 - 1/p')}\, \|u_0\|_1.$$

As we have taken $\frac{N}{2} < p$ the last term of the inequality (59) is bounded and it tends to 0 as $\delta \to 0$ uniformly in $\lambda > \lambda_0 > 0$.

Using that

$$\int_{\mathbb{R}^N} u_\lambda(0, x)\, \psi_2(x)\, dx \to M \psi_2(0) \qquad \text{as} \quad \lambda \to \infty$$

we can conclude that for any $\varepsilon > 0$ there exist some $\tau > 0$ and $\lambda_0 > 1$ such that

$$\left| \int_{\mathbb{R}^N} u_\lambda(\delta, x)\, \psi_2(x)\, dx - M\psi_2(0) \right| < \varepsilon$$

if $0 < \delta < \tau$ and $\lambda > \lambda_0$. Therefore, we have

$$\lim_{\delta \to 0^+} \lim_{\lambda \to \infty} \int_{\mathbb{R}^N} u_\lambda(\delta, x)\, \psi_2(x)\, dx = M\psi_2(0) \tag{61}$$

for every $\psi_2 \in C_c^\infty(\mathbb{R}^N)$. Thus, (56) and consequently (53) hold for every $\psi_2 \in C_c^\infty(\mathbb{R}^N)$. By estimate (41), we have that (53) holds for every $\psi_2 \in BC^2(\mathbb{R}^N)$.

Therefore if we pass to the limit as $\delta \to 0$ in (52) we obtain that u^* satisfies

$$\int_0^T \int_{\mathbb{R}^N} (a^h\, \nabla u^* \cdot \nabla \psi - u^* \psi_t)\, dx\, dt = M\psi(0, 0) - \int_{\mathbb{R}^N} u^*(T)\, \psi(T)\, dx \tag{62}$$

for every $\psi(t, x) = \psi_1(t)\, \psi_2(x)$ with $\psi_1 \in W^{1, \infty}[0, T]$ and $\psi_2 \in BC^2(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$.

We deduce that u^* is a weak solution of problem (6).

Step 4. As a consequence of the above step we can conclude that there exists a subsequence u_λ (that we still denote by the index λ to simplify the notation), such that

$$u_\lambda \rightharpoonup u^* \qquad \text{weakly in} \quad L^2_{loc}(0, \infty; H^1(\mathbb{R}^N))$$

as $\lambda \to \infty$, where $u^*(x, t)$ is a weak solution of the problem (6). The uniqueness of the weak solution of problem (6) guarantees that $u^* = u^h$, and forces the whole family u_λ to converge to u^h. On the other hand, the compactness result (50) allows to deduce that the convergence holds in $C([t_1, t_2]; L^2(\Omega))$.

Therefore we have

$$u_\lambda(t) \to u^h(t) \qquad \text{in} \quad L^1(B_R)$$

as $\lambda \to \infty$, for every $t \in [t_1, t_2]$.

By (41), we conclude that (23) holds for $p = 1$.

Combining the fact that $u_\lambda(1) \to u^h(1)$ strongly in $L^1(IR^N)$ as $\lambda \to \infty$ and

$$\|u_\lambda(1)\|_\infty \leqslant C_\infty\, \|u_0\|_1,$$

which is a consequence of (4) with $p=\infty$, we deduce that $u_\lambda(1)\to u^h(1)$ strongly in $L^p(\mathbb{R}^N)$ as $\lambda\to\infty$ for every $1\leqslant p<\infty$. Therefore (23) holds for every $p\in[1,\infty)$. Finally we see that for dimension $N=1$ and $N\geqslant 2$, (23) follows for $p=\infty$.

Indeed, thanks to Gagliardo-Nirenberg inequality we have

$$\|u(t)-MG(t)\|_\infty \leqslant C\,\|u(t)-MG(t)\|_p^{1/2}\,\|\nabla u(t)-M\,\nabla G(t)\|_q^{1/2} \tag{63}$$

with $\frac{1}{p}+\frac{1}{q}=\frac{1}{N}$.

For dimension $N=1$, by (40) we deduce

$$\|\nabla u(t)-M\,\nabla G(t)\|_2^{1/2}\leqslant Ct^{-3/8}.$$

Combining this result with (63) for $p=q=2$ we can conclude that

$$t^{1/2}\,\|u(t)-MG(t)\|_\infty\leqslant t^{(1/4)(1-1/2)}\,\|u(t)-MG(t)\|_2^{1/2}.$$

Consequently (23) holds for $N=1$ and $p=\infty$.

For dimension $N=2$, by (25) we have

$$\|\nabla u(t)-M\,\nabla G(t)\|_r^{1/2}\leqslant Ct^{1/2r-3/4}$$

for any $2<r<r^*$.

Therefore taking $p=r$ and $q=\frac{1}{N}-\frac{1}{r}$ in (63) we deduce that

$$t\|u(t)-MG(t)\|_\infty\leqslant t^{(1/2)(1-(r-2)/2r)}\,\|u(t)-MG(t)\|_{2r/(r-2)}^{1/2}.$$

Consequently (23) holds for $N=2$ and $p=\infty$.

This concludes the proof of Theorem 1. ∎

4. WEAKLY NONLINEAR BEHAVIOR

In this section we study the asymptotic behavior of the solution of the problem

$$\begin{cases} u_t-\operatorname{div}(a(x)\,\nabla u)=d\cdot\nabla(|u|^{q-1}\,u) & \text{in}\quad (0,\infty)\times\mathbb{R}^N\\ u(0,x)=u_0(x) \end{cases} \tag{64}$$

with $q>1+\frac{1}{N}$, $N\geqslant 1$ and initial data $u_0\in L^1(\mathbb{R}^N)\cap L^r(\mathbb{R}^N)$ with $r>\frac{N}{2}(q-1)$. We assume that the coefficients a_{ij} satisfy all the assumptions of the introduction.

We prove that the large time behavior of solutions of the problem (64) is given by the fundamental solution of the equation

$$\begin{cases} u_t^h - \operatorname{div}(a^h \nabla u^h) = 0 \\ u^h(0, x) = M\delta, \end{cases} \tag{65}$$

where a^h is the homogenized matrix.

More precisely, we have the following result:

THEOREM 2. *Let $q > 1 + \frac{1}{N}$, with $N \geqslant 1$. Then, for every $u_0 \in L^1(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ with $r > \frac{N}{2}(q-1)$ such that $M = \int_{\mathbb{R}^N} u_0(x)\, dx$ the solution of* (64) *satisfies*

$$t^{(N/2)(1-1/p)} \, \|u(t) - u^h(t)\|_p \to 0 \qquad \text{as} \quad t \to \infty \tag{66}$$

for every $p \in [1, \infty)$, where u^h is the solution of (65).

Moreover, if $N = 1$, the solution $u = u(t, x)$ satisfies (66) *also for $p = \infty$.*

Remark 3. We observe that when $q > 1 + \frac{2}{N}$ the hypothesis in this Theorem over the initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ with $r > \frac{N}{2}(q-1)$ is more restrictive than the hypothesis $r > \frac{Nq}{N+2}$ used in the variational formulation (2). We need this stronger hypothesis to obtain the uniform estimates of the scaled solution u_λ as $|x| \to \infty$ (see Proposition 4 below).

Proof of the Theorem 2.

Step 1. Let u be the solution of (64). Then the scaled functions $u_\lambda(t, x) = \lambda^N u(\lambda^2 t, \lambda x)$ solve

$$\begin{cases} u_{\lambda, t} - \operatorname{div}(a(\lambda x) \nabla u_\lambda) = \lambda^{N(1-q)+1} \, d \cdot \nabla(|u_\lambda|^{q-1} u_\lambda) \qquad \text{in} \quad (0, \infty) \times \mathbb{R}^N \\ u_\lambda(0, x) = u_{\lambda, 0}(x) = \lambda^N u_0(\lambda x). \end{cases} \tag{67}$$

On the hand, as u^h is self-similar, proving (66) is equivalent to prove that

$$u_\lambda(t_0) \to u^h(t_0) \qquad \text{in} \quad L^p(\mathbb{R}^N) \qquad \text{as} \quad \lambda \to \infty$$

for every $p \in [1, \infty)$ and for some $t_0 > 0$ fixed. Thus, the proof of the convergence result in Theorem 2 consists precisely in showing that u_λ converges to the solution u^h of equation (65) at time $t = 1$ when $\lambda \to \infty$.

Step 2: Uniform estimates for the scaled solutions. Multiplying in (64) by powers of u and integrating by parts as in [13] it follows that

$$\|u(t)\|_p \leqslant C_p \|u_0\|_1 t^{(-N/2)(1-1/p)} \qquad \forall t>0 \tag{68}$$

for every $p \in [1, \infty]$ and $C_p > 0$.

As a consequence of (68) we get

$$\|u_\lambda(t)\|_p \leqslant C_p \|u_0\|_1 t^{(-N/2)(1-1/p)}, \qquad \forall t>0 \tag{69}$$

for every $p \in [1, \infty]$ with $C_p > 0$ independent of λ. By (69) we deduce that

$$\| |u_\lambda|^q (t)\|_p \leqslant C_{pq} \|u_0\|_1^q t^{(-qN/2)(1-1/pq)}, \qquad \forall t>0 \tag{70}$$

for every $p \in [1, \infty]$ and with $C_{pq} > 0$ independent of λ.

On the other hand, multiplying by u_λ in the equation (67), integrating in $(t_1, t_2) \times \mathbb{R}^N$ and integrating by parts it follows that

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 \, dx \, dt \leqslant C \left(\int_{\mathbb{R}^N} u_\lambda^2(t_1, x) \, dx - \int_{\mathbb{R}^N} u_\lambda^2(t_2, x) \, dx \right) \leqslant C \tag{71}$$

uniformly on $\lambda > 0$ if $0 < t_1 < t_2 < \infty$.

Now, we need the following Lemma:

LEMMA 2. *Let u_λ be the solution of problem* (67). *Then, for each $0 < \tau < T$ and $0 \leqslant s < 1$, u_λ is uniformly bounded in $L^\infty((\tau, T); H^s(\mathbb{R}^N))$ for $\lambda \geqslant 1$.*

Proof of Lemma 2. In view of (69) and (71) the scaled solutions satisfy

$$\begin{cases} u_{\lambda, t} - \operatorname{div}(a(\lambda x) \nabla u_\lambda) = f_\lambda & \text{in} \quad (0, \infty) \times \mathbb{R}^N \\ u_\lambda(\tau) = u_{\lambda, \tau}(x) \end{cases} \tag{72}$$

with $f_\lambda = \lambda^{N(1-q)+1} d \cdot \nabla(|u_\lambda|^{q-1} u_\lambda)$ bounded in $L^2((\tau, T) \times \mathbb{R}^N)$ and $u_\lambda(\tau)$ bounded in all the spaces L^p, $1 \leqslant p \leqslant \infty$, for every $\tau > 0$.

Let T_λ be the semigroup associated to (72). According to the result of section 3 above it follows

$$\|\nabla T_\lambda(t) u_0\|_2 \leqslant C t^{-1/2} \|u_0\|_2, \qquad \forall t > 0$$

$$\|T_\lambda(t) u_0\|_2 \leqslant \|u_0\|_2, \qquad \forall t > 0$$

with $C > 0$ independent of λ.

Then, by interpolation

$$\| T_\lambda(t)\, u_0 \|_{H^s} \leqslant C t^{-s/2} \, \|u_0\|_2, \qquad \forall t > 0 \tag{73}$$

if $0 \leqslant s \leqslant 1$, with $C > 0$ independent of λ.

The solution u_λ of (72) verifies the integral equation

$$u_\lambda(t) = T_\lambda(t-\tau)\, u_\lambda(\tau) + \int_0^{t-\tau} T(t-\tau-\sigma)\, f_\lambda(\sigma+\tau)\, d\sigma.$$

Taking H^s norms and applying (73) we have that

$$\begin{aligned}\|u_\lambda(t)\|_{H^s} &\leqslant C(t-\tau)^{-s/2} \, \|u_\lambda(\tau)\|_2 + C \int_0^{t-\tau} (t-\tau-\sigma)^{-s/2} \, \|f_\lambda\|_2 \, d\sigma \\ &\leqslant C(t-\tau)^{-s/2} + C \, \|f_\lambda\|_{L^2((\tau,\, T) \times \mathbb{R}^N)} \left(\int_0^{t-\tau} (t-\tau-\sigma)^{-s} \, d\sigma \right)^{1/2}.\end{aligned}$$

The last integral is bounded for $t \in [\tau, T]$ if $s < 1$. Therefore, if $0 \leqslant s < 1$ we obtain that $\|u_\lambda(t)\|_{H^s}$ is uniformly bounded in $\lambda \geqslant 1$ for any $t > 0$ fixed. The proof of Lemma 2 is now completed.

By (69) and (71) we deduce that

(a) $\{u_\lambda\}$ is uniformly bounded in $L^2((t_1, t_2); H^1(\mathbb{R}^N))$ for every $0 < t_1 < t_2 < \infty$.

Using (70), (71), the fact that $a(\lambda x)$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$ and equation (67) we deduce that

(b) $\{\partial_t u_\lambda\}$ is uniformly bounded in $L^2_{loc}((0, \infty); H^{-1}(\Omega))$ for every bounded domain Ω of $\mathbb{R}^N$.

On the other hand, as a consequence of (69) we have

(c) $\{u_\lambda\}$ is uniformly bounded in $L^\infty_{loc}((0, \infty); L^2_{loc}(\mathbb{R}^N))$.

Taking into account that $L^2(\Omega)$ is compactly embedded in $H^{-\varepsilon}(\Omega)$ for every $\varepsilon > 0$, and that $H^{-\varepsilon}(\Omega) \hookrightarrow H^{-1}(\Omega)$ with continuous embedding if $0 < \varepsilon < 1$, combining (b) and (c) and applying classical compactness results (cf. [20], Corollary 4, p. 85) we deduce that

(d) $\{u_\lambda\}$ is relatively compact in $C([t_1, t_2]; H^{-\varepsilon}(\Omega))$ for every $0 < t_1 < t_2 < \infty$ and $\varepsilon > 0$.

Extracting a subsequence $\lambda_n \to \infty$ (that we still denote by λ), we may assert that

$$u_\lambda \to u^* \qquad \text{in} \quad C([t_1, t_2]; H^{-\varepsilon}(\Omega)) \tag{74}$$

for every bounded domain Ω and every $\varepsilon > 0$ with $0 < t_1 < t_2 < \infty$. On the other hand, as a consequence of Lemma 2, we conclude that $u_{\lambda_n}(t)$ is relatively compact in $L^2_{loc}(IR^N)$ for every $t \in [t_1, t_2]$. By (74) we have

$$u_{\lambda_n}(t) \to u^*(t) \qquad \text{in} \quad L^2_{loc}(\mathbb{R}^N) \qquad \text{as} \quad \lambda_n \to \infty$$

for every $t \in [t_1, t_2]$. Therefore, taking (69) into account we deduce

$$u_{\lambda_n}(t) \to u^*(t) \qquad \text{in} \quad L^p_{loc}(\mathbb{R}^N) \qquad \text{as} \quad \lambda_n \to \infty$$

for every $t \in [t_1, t_2]$ and $1 \leqslant p < \infty$.

Step 3: *Uniform estimates on u_λ as $|x| \to \infty$.*

PROPOSITION 4. *Let u_λ be the solution of problem* (67). *Then*

$$\|u_\lambda(t, \cdot)\|_{L^1(|x| > R)} \to 0 \qquad \textit{as} \quad R \to \infty \tag{75}$$

uniformly in $\lambda \geqslant 1$ and $t \in [0, t_0]$, for any $t_0 > 0$ fixed.

Proof of Proposition 4. Let u_λ, $\bar{u}_\lambda$ and $\underline{u}_\lambda$ be the solutions of

$$u_{\lambda, t} - \operatorname{div}(a(\lambda x) \nabla u_\lambda) = \lambda^{N(1-q)+1} d \cdot \nabla(|u_\lambda|^{q-1} u_\lambda) \text{ in } (0, \infty) \times \mathbb{R}^N \tag{76}$$

with initial data $u_\lambda(0)$, $|u_\lambda|(0)$ and $-|u_\lambda|(0)$ respectively.

Then as $-|u_\lambda|(0) \leqslant u_\lambda(0) \leqslant |u_\lambda|(0)$, applying Lemma 1 of [9] and taking into account that $\underline{u}_\lambda = -\bar{u}_\lambda$ we deduce

$$|u_\lambda| \leqslant \bar{u}_\lambda \qquad \forall t > 0, \quad \text{a.e.} \quad x \in \mathbb{R}^N.$$

Therefore, we have

$$\int_{|x| > R} |u_\lambda|(t, x)\, dx \leqslant \int_{|x| > R} \bar{u}_\lambda(t, x)\, dx.$$

To prove (75) it is sufficient to obtain that

$$\int_{|x| > R} \bar{u}_\lambda(t, x)\, dx \to 0 \qquad \text{as} \quad R \to \infty$$

uniformly in $\lambda \geqslant 1$ and $t \in [0, t_0]$. We consider the equation

$$\bar{u}_{\lambda, t} - \operatorname{div}(a(\lambda x) \nabla \bar{u}_\lambda) = \lambda^{N(1-q)+1} d \cdot \nabla(|\bar{u}_\lambda|^{q-1} u_\lambda) \qquad \text{in} \quad (0, \infty) \times \mathbb{R}^N$$

that $\bar{u}_\lambda$ satisfies.

We define a radial function $\varphi \in BC^2(\mathbb{R}^N)$ such that

$$\varphi(r) = \begin{cases} 0, & 0 < r < \frac{1}{2} \\ 1, & r > 1 \end{cases}$$

and $0 \leqslant \varphi \leqslant 1$ when $\frac{1}{2} < r < 1$. Then we set $\varphi_R(x) = \varphi(x/R)$ for every $R > 0$.

Multiplying the equation (76) by φ_R, integrating by parts and taking (58) into account it follows that

$$\begin{aligned} \frac{d}{dt} \int \bar{u}_\lambda(t, x)\, \varphi_R(x)\, dx = \int \bar{u}_\lambda(t, x)\, \frac{1}{R^2}\, a_{ij}(\lambda x)\, \partial^2_{ij} \varphi\left(\frac{x}{R}\right) dx \\ + \int \bar{u}_\lambda(t, x)\, \lambda\, \frac{1}{R}\, \partial_i a_{ij}(\lambda x)\, \partial_j \varphi\left(\frac{x}{R}\right) \nabla\varphi\left(\frac{x}{R}\right) dx \\ - \lambda^{N(1-q)+1} \int |\bar{u}_\lambda|^{q-1}\, \bar{u}_\lambda\, \frac{d \cdot \nabla\varphi(x/R)}{R}\, dx \end{aligned}$$

and therefore

$$\begin{aligned} &\frac{d}{dt} \int \bar{u}_\lambda(t, x)\, \varphi_R(x)\, dx \\ &\quad \leqslant \frac{1}{R^2} \left\| a_{ij}(\lambda x)\, \partial^2_{ij} \varphi\left(\frac{x}{R}\right) \right\|_\infty \|\bar{u}_\lambda\|_1 \\ &\quad + \frac{1}{R} \big[\|\lambda \operatorname{div}(a(\lambda x))\|_p\, \|\nabla\varphi\|_\infty\, \|\bar{u}_\lambda\|_{p'} \\ &\quad + \lambda^{N(1-q)+1}\, |d|\, C\, \|\nabla\varphi\|_\infty\, \|\bar{u}_\lambda\|_\infty^{q-1}\, \|\bar{u}_\lambda\|_1 \big]. \end{aligned}$$

Integrating in the time interval $[0, t]$ we have

$$\begin{aligned} \int \bar{u}_\lambda(t, x)\, \varphi_R(x)\, dx \leqslant \int \bar{u}_\lambda(0, x)\, \varphi_R(x)\, dx \\ + \frac{1}{R^2} \int_0^t \left\| a_{ij}(\lambda x)\, \partial^2_{ij} \varphi\left(\frac{x}{R}\right) \right\|_\infty \|\bar{u}_\lambda\|_1\, dt \\ + \frac{1}{R} \int_0^t \big[\|\lambda \operatorname{div}(a(\lambda x))\|_p\, \|\nabla\varphi\|_\infty\, \|\bar{u}_\lambda\|_{p'} \\ + \lambda^{N(1-q)+1}\, |d|\, C\, \|\nabla\varphi\|_\infty\, \|\bar{u}_\lambda\|_\infty^{q-1}\, \|\bar{u}_\lambda\|_1 \big]\, dt. \end{aligned} \tag{77}$$

In view of (69) and taking into account that $a(\lambda x)$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$ we deduce that

$$\frac{1}{R^2}\left\|a_{ij}(\lambda x)\,\partial_{ij}^2\varphi\left(\frac{x}{R}\right)\right\|_\infty \int_0^t \|\bar{u}_\lambda\|_1\,dt \leqslant \frac{C}{R^2}\,t.$$

Taking $\frac{N}{2} < p \leqslant N$ (like in step 3 of in the proof of Theorem 1) we obtain that

$$\frac{1}{R}\,\|\lambda\,\mathrm{div}(a(\lambda x))\|_p\,\|\nabla\varphi\|_\infty \int_0^t \|\bar{u}_\lambda\|_{p'}\,dt \leqslant \frac{C}{R}\,t^\epsilon$$

for some $\epsilon > 0$. Finally we are going to estimate the last term in (77). We distinguish two cases. When $1+\frac{1}{N} < q < 1+\frac{2}{N}$ we obtain that

$$\begin{aligned}
&\frac{C\lambda^{N(1-q)+1}}{R}\,\|\nabla\varphi\|_\infty \int_0^t \|\bar{u}_\lambda\|_\infty^{q-1}\,\|\bar{u}_\lambda\|_1\,dt \\
&\qquad\leqslant \frac{C\lambda^{N(1-q)+1}}{R}\int_0^t \|\bar{u}_\lambda\|_\infty^{q-1}\,dt \\
&\qquad\leqslant \frac{C\lambda^{N(1-q)+1}}{R}\int_0^t s^{(N/2)(1-q)}\,dt \leqslant \frac{C\lambda^{N(1-q)+1}}{R}\,t^{(N/2)(1-q)+1}.
\end{aligned}$$

As $q \geqslant 1+\frac{1}{N}$, the power of λ is non positive.

In the case $q \geqslant 1+\frac{2}{N}$ the argument above fails since $s^{(N/2)(1-q)}$ does not belong to $L^1(0, t)$ and we use the hypothesis $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ with $r > \frac{N}{2}(q-1)$. By classical $L^r - L^p$ estimates (see Proposition 1 of [9]) we have

$$\|u(t)\|_p \leqslant C\,\|u_0\|_r\,t^{(-N/2)(1/r-1/p)}, \qquad \forall t > 0$$

for every $1 \leqslant r \leqslant p \leqslant \infty$. Taking $p = \infty$ we have

$$\|u(t)\|_\infty \leqslant Ct^{-N/2r}\,\|u_0\|_r, \qquad \forall t > 0.$$

Thus, u_λ satisfies

$$\|u_\lambda(t)\|_\infty \leqslant Ct^{-N/2r}\lambda^{N(1-1/r)}\,\|u_0\|_r \qquad \forall t > 0. \tag{78}$$

Therefore, we obtain

$$\int_0^t \|\bar{u}_\lambda\|_\infty^{q-1}\,dt \leqslant \frac{C\lambda^{1-N(q-1)/r}}{R}\int_0^t s^{(-N/2r)(q-1)}\,dt \leqslant \frac{C\lambda^{1-N(q-1)/r}}{R}\,t^{(1-N)/2r(q-1)}$$

if we take $r > \frac{N(q-1)}{2}$. Moreover, we also need to see that the power of λ is non positive. For this we choose $r \leqslant N(q-1)$ so that $1 - \frac{N(q-1)}{r} \leqslant 0$.

Therefore, if $0 \leqslant t \leqslant t_0$, we have

$$\int_{|x|>R} \bar{u}_\lambda(t, x)\, dx \leqslant \int \bar{u}_\lambda(t, x)\, \varphi_R(x)\, dx \leqslant \int \bar{u}_\lambda(0, x)\, \varphi_R(x)\, dx + \frac{C}{R^2}\, t_0 + \frac{C}{R}\, t_0^{\epsilon} + \frac{C}{R},$$

with $q > 1 + \frac{1}{N}$. Thus, it is enough to see that

$$\int \bar{u}_\lambda(0, x)\, \varphi_R(x) \to 0 \qquad \text{as} \quad R \to \infty \tag{79}$$

uniformly for $\lambda > 1$. But this is immediate since

$$\begin{aligned}\int \bar{u}_\lambda(0, x)\, \varphi_R(x)\, dx &= \int \lambda^N \bar{u}(0, \lambda x)\, \varphi\left(\frac{x}{R}\right) dx \\ &\leqslant \int_{|x|>R/2} \lambda^N \bar{u}(0, \lambda x)\, dx = \int_{|y|>\lambda R/2} \bar{u}(0, y)\, dy \to 0, \qquad \text{as} \quad R \to \infty\end{aligned}$$

uniformly on $\lambda > 1$, since $\bar{u}(0) \in L^1(\mathbb{R}^N)$. Therefore (79) holds. The proof of Proposition 4 is now completed. ∎

Step 4: *Passage to the limit in the variational formulation.* Let $\psi \in W^{1,1}([0, T]; BC^2(\mathbb{R}^N) \cap H^2(\mathbb{R}^N))$ be a test function. By the variational formulation we have

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^N} (a(\lambda x)\, \nabla u_\lambda \cdot \nabla \psi - u_\lambda \psi_t)\, dx\, dt &= \int_{\mathbb{R}^N} u_{0,\lambda} \psi(0)\, dx - \int_{\mathbb{R}^N} u_\lambda(T)\, \psi(T)\, dx \\ &\quad - \lambda^{N(1-q)+1} \int_0^T \int_{\mathbb{R}^N} d \cdot \nabla \psi(x) |u_\lambda|^{q-1} u_\lambda\, dx\, dt.\end{aligned}$$

We consider $\delta > 0$ such that $\delta < T$. We also have

$$\int_\delta^T \int_{\mathbb{R}^N} (a(\lambda x) \nabla u_\lambda \cdot \nabla \psi - u_\lambda \psi_t) \, dx \, dt$$

$$+ \lambda^{N(1-q)+1} \int_0^T \int_{\mathbb{R}^N} d \cdot \nabla \psi(x) \, |u_\lambda|^{q-1} u_\lambda \, dx \, dt$$

$$= \int_{\mathbb{R}^N} u_\lambda(\delta) \, \psi(\delta) \, dx - \int_{\mathbb{R}^N} u_\lambda(T) \, \psi(T) \, dx. \tag{80}$$

We are going to pass to the limit twice in (80). First as $\lambda \to \infty$ and then as $\delta \to 0$.

We proceed as in the linear case. We consider test functions of the form $\psi(t, x) = \psi_1(t) \, \psi_2(x)$ with $\psi_1(t) \in W^{1,\infty}[0, T]$ and $\psi_2(x) \in BC^2(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$, and consider $\hat{u}_\lambda$ and $\check{u}_\lambda$ as in the proof of Theorem 2.

Since u_λ satisfies (80) we have that $\hat{u}_\lambda$ satisfies

$$\int_{\mathbb{R}^N} [a(\lambda x) \nabla \hat{u}_\lambda \cdot \nabla \psi_2 + \check{u} \psi_2] \, dx$$

$$= -\psi_1(T) \int_{\mathbb{R}^N} u_\lambda(T) \, \psi_2 \, dx + \psi_1(\delta) \int_{\mathbb{R}^N} u_\lambda(\delta) \, \psi_2 \, dx$$

$$+ \lambda^{N(1-q)+1} \int_0^T \int_{\mathbb{R}^N} d \cdot \psi_2 \psi_1 \nabla(|u_\lambda|^{q-1} u_\lambda) \, dx \, dt \equiv I_1 + I_2 + I_3. \tag{81}$$

Therefore we have that $\hat{u}_\lambda$ satisfies

$$-\operatorname{div}(a_\lambda \nabla \hat{u}_\lambda) + \hat{u}_\lambda = f_\lambda \qquad \text{in} \quad \mathbb{R}^N,$$

where

$$f_\lambda = -\check{u}_\lambda + \hat{u}_\lambda - u_\lambda(T) \, \psi_1(T) + u_\lambda(\delta) \, \psi_1(\delta)$$

$$+ \lambda^{N(1-q)+1} \int_0^T d \cdot \psi_1 \, \nabla(|u_\lambda|^{1/N} u_\lambda) \, dt.$$

We claim that f_λ is bounded in $H^{-1}(\mathbb{R}^N)$ and that (46) holds for all bounded domain Ω of $\mathbb{R}^N$ with $f = -\check{u}^* + \hat{u}^* - u^*(T) \, \psi_1(T) + u^*(\delta) \, \psi_1(\delta)$. By the Remark 1 we then deduce that $\hat{u}^*$ satisfies

$$-\operatorname{div}(a^h \nabla \hat{u}^*) + \hat{u}^* = -\check{u}^* + \hat{u}^* - u^*(T) \, \psi_1(T) + u^*(\delta) \, \psi_1(\delta) \qquad \text{in} \quad \mathbb{R}^N.$$

First, since $\{u_\lambda\}$ is uniformly bounded in $L^2((t_1, t_2); H^1(\mathbb{R}^N))$ for every $0 < t_1 < t_2 < \infty$ by extracting subsequences, we have that (43)–(44) hold. As in the linear case, one can also pass to the limit in I_1 and I_2.

Finally, integrating by parts in the last term of f_λ we claim that

$$\lambda^{N(1-q)+1} \int_0^T d \cdot \nabla\psi_1 \, |u_\lambda|^{q-1} u_\lambda \, dt \to 0 \quad \text{in} \quad H^{-1}(\Omega) \quad \text{as} \quad \lambda \to \infty. \tag{82}$$

In order to prove this fact we distingish two cases. First, we suppose that $1 + \frac{1}{N} < q < 1 + \frac{2}{N}$. In this case, we have that

$$\begin{aligned} &\lambda^{N(1-q)+1} \left| \int_0^T \int_{\mathbb{R}^N} d \cdot \nabla\psi(t, x) \, |u_\lambda|^{q-1} u_\lambda \, dx \, dt \right| \\ &\quad \leqslant C \, |d| \, \lambda^{N(1-q)+1} \int_0^T \| |u_\lambda|^q (s) \|_1 \, ds \leqslant C \, |d| \, \lambda^{N(1-q)+1} \int_0^T \| |u_\lambda|(s) \|_q^q \, ds \\ &\quad \leqslant C \, \|u_0\|_1^q \, \lambda^{N(1-q)+1} \int_0^T s^{(N/2)(1-q)} \leqslant C \lambda^{N(1-q)+1} T^{(N/2)(1-q)+1}. \end{aligned} \tag{83}$$

Therefore (82) holds when $1 + \frac{1}{N} < q < 1 + \frac{2}{N}$.

If $q \geqslant 1 + \frac{2}{N}$ we have that $u_0 \in L^1(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ with $r > \frac{N(q-1)}{2} \geqslant 1$. In this case, we also have the $L^r - L^\infty$ estimate (78). Therefore, repeating the argument of Proposition 4, we have that there exists some $p \geqslant \frac{N}{2}(q-1)$ such that

$$\begin{aligned} \lambda^{N(1-q)+1} \int_0^t \|u_\lambda\|_\infty^{q-1} \, dt &\leqslant \frac{C \lambda^{1 - N(q-1)/p}}{R} \int_0^t s^{(-N/2p)(q-1)} \, dt \\ &\leqslant \frac{C \lambda^{1 - N(q-1)/p}}{R} \, t^{(1 - N/2p)(q-1)} \end{aligned} \tag{84}$$

with $1 - \frac{N(q-1)}{p} \leqslant 0$. Therefore, (82) holds when $q \geqslant 1 + \frac{2}{N}$.

We deduce that u^* satisfies:

$$\begin{aligned} &\int_{\mathbb{R}^N} [a^h \, \nabla\hat{u}^* \cdot \nabla\psi_2 + \check{u}^* \psi_2] \, dx \\ &\quad = -\int_{\mathbb{R}^N} u^*(T) \, \psi_2 \psi_1(T) \, dx + \int_{\mathbb{R}^N} u^*(\delta) \, \psi_2 \psi_1(\delta) \, dx. \end{aligned} \tag{85}$$

We also have (54) for every $\psi_2 \in BC^2(\mathbb{R}^N)$. Therefore it is enough to prove (56) for every $\psi_2 \in BC^2(\mathbb{R}^N)$. Multiplying the equation (9) by a test function $\psi_2 \in C_c^\infty(\mathbb{R}^N)$ and integrating in $(0, \delta) \times \mathbb{R}^N$ we have:

$$\left| \int_{\mathbb{R}^N} u_\lambda(\delta, x)\, \psi_2(x) - \int_{\mathbb{R}^N} u_\lambda(0, x)\, \psi_2(x) \right|$$

$$\leqslant \left| \int_0^\delta \int_{\mathbb{R}^N} a(\lambda x)\, \nabla u_\lambda(s, x) \cdot \nabla \psi_2(x)\, dx\, ds \right|$$

$$+ \lambda^{N(1-q)+1} \left| d \cdot \int_0^\delta \int_{\mathbb{R}^N} |u_\lambda(s, x)|^{q-1}\, u_\lambda(s, x)\, \nabla \psi_2(x)\, dx\, ds \right| \equiv |I_1| + |I_2|.$$

By (83) and (84) we deduce that for any $\varepsilon > 0$ there exist some $\tau > 0$ and $\lambda_0 > 1$ such that $|I_2| < \varepsilon$ if $0 < \delta < \tau$ and $\lambda > \lambda_0$.

Repeating the arguments of the linear case we can deduce that for any $\varepsilon > 0$ there exist some $\tau > 0$ and $\lambda_0 > 1$ such that $|I_1| < \varepsilon$ if $0 < \delta < \tau$ and $\lambda > \lambda_0$. Therefore taking into account these estimates and that

$$\int_{\mathbb{R}^N} u_\lambda(0, x)\, \psi_2(x)\, dx \to M\psi_2(0) \qquad \text{as} \quad \lambda \to \infty$$

we conclude that for any $\varepsilon > 0$ there exist $\tau > 0$ and $\lambda_0 > 1$ such that

$$\left| \int_{\mathbb{R}^N} u_\lambda(\delta, x)\, \psi_2(x)\, dx - M\psi_2(0) \right| < \varepsilon$$

if $0 < \delta < \tau$ and $\lambda > \lambda_0$. Therefore we conclude, as in the linear case, that

$$\lim_{\delta \to 0^+} \int_{\mathbb{R}^N} u^*(\delta, x)\, \psi_2(x)\, dx = M\psi_2(0) \tag{86}$$

for every $\psi_2 \in C_c^\infty(\mathbb{R}^N)$. By Proposition 4 we deduce that (86) is satisfied for every $\psi_2 \in BC^2(\mathbb{R}^N)$.

Therefore we obtain that u^* is a weak solution of problem (65). Thanks to the uniqueness of solution of problem (65) we have $u^* = u^h$ and we conclude that the above limit is satisfied by the whole family u_λ. Repeating the arguments of the step 4 of Theorem 1 we can conclude that (66) is satisfied for every $1 \leqslant p < \infty$.

Finally we see that for dimension $N = 1$, (66) follows for $p = \infty$.

Indeeds, thanks to Gagliardo-Niremberg inequality we have

$$\|u(t) - u^h(t)\|_\infty \leqslant C\, \|u(t) - u^h(t)\|_6^{1/2}\, \|D^{2/3}u(t) - D^{2/3}u^h(t)\|_2^{1/2}.$$

By Lemma 2 we have

$$\|D^{2/3}u(t) - D^{2/3}u^h(t)\|_2^{1/2} \leqslant Ct^{-7/24}.$$

Therefore

$$t^{1/2}\,\|u(t) - u^h(t)\|_\infty \leqslant t^{1/4(1-1/6)}\,\|u(t) - u^h(t)\|_6^{1/2}.$$

Consequently (66) holds for $N=1$ and $p=\infty$.

This concludes the proof of Theorem 2.

5. SELF-SIMILAR BEHAVIOR

In this section we study the asymptotic self-similar behavior of the solution of the problem

$$\begin{cases} u_t - \operatorname{div}(a(x)\,\nabla u) = d \cdot \nabla(|u|^{1/N}\,u) & \text{in} \quad (0,\infty)\times\mathbb{R}^N \\ (0,x) = u_0 \in L^1(\mathbb{R}^N) \end{cases} \tag{87}$$

with $N \geqslant 1$. We suppose that the coefficients a_{ij} satisfy all the assumptions of the introduction.

We prove that the large time behavior of solutions of (87) is given by solutions of equation

$$\begin{cases} u_t^h - \operatorname{div}(a^h\,\nabla u^h) = d \cdot \nabla(|u^h|^{1/N}\,u^h) & \text{in} \quad (0,\infty)\times\mathbb{R}^N \\ u^h(0,x) = M\delta, \end{cases} \tag{88}$$

where a^h is the homogenized matrix. More precisely:

THEOREM 3. *Let $u_0 \in L^1(\mathbb{R}^N)$ with $M = \int_{\mathbb{R}^N} u_0(x)\,dx$. Then the solution of* (87) *satisfies*

$$t^{(N/2)(1-1/p)}\,\|u(t) - u^h(t)\|_p \to 0 \qquad \text{as} \quad t\to\infty \tag{89}$$

for every $p\in[1,\infty)$, where $u^h(t)$ is the unique solution of the equation (88). *This solution u^h has a self-similar structure*: *$u^h(t,x) = t^{-N/2}f(x/\sqrt{t})$ with $\int_{\mathbb{R}^N} f(x)\,dx = M$. Moreover, if $N=1$,* (89) *holds also for $p=\infty$.*

Proof of Theorem 3.

Step 1. Let u be the solution of (87). Then the scaled fuctions $u_\lambda(t,x) = \lambda^N u(\lambda^2 t, \lambda x)$ solve

$$\begin{cases} u_{\lambda,t} - \operatorname{div}(a(\lambda x)\,\nabla u_\lambda) = d \cdot \nabla(|u_\lambda|^{1/N}\,u_\lambda) & \text{in} \quad (0,\infty)\times\mathbb{R}^N \\ u_\lambda(0,x) = u_{\lambda,0}(x) = \lambda^N u_0(\lambda x). \end{cases} \tag{90}$$

On the other hand, as u^h is self-similar we observe that proving (89) is equivalent to prove

$$u_\lambda(1) \to u^h(1) \qquad \text{in} \quad L^p(\mathbb{R}^N) \qquad \text{as} \quad \lambda \to \infty$$

for every $p \in [1, \infty)$.

The compactness arguments of step 2 of the proof of Theorem 2 allow to show that we can extract a subsequence $\lambda_n \to \infty$ such that

$$u_{\lambda_n} \to u^* \qquad \text{strongly in} \quad C([t_1, t_2]; H^{-\varepsilon}(\Omega)) \tag{91}$$

for every bounded domain Ω and every $\varepsilon > 0$ with $0 < t_1 < t_2 < \infty$ and that

$$u_{\lambda_n}(t) \to u^*(t) \qquad \text{in} \quad L^p_{loc}(\mathbb{R}^N) \qquad \text{as} \quad \lambda_n \to \infty \tag{92}$$

for every $t \in [t_1, t_2]$ and $1 \leqslant p < \infty$.

Step 2: *Passage to the limit in the variational formulation.* In this case, $\hat{u}_\lambda$ verifies

$$-\operatorname{div}(a_\lambda \nabla \hat{u}_\lambda) + \hat{u}_\lambda = f_\lambda \qquad \text{in} \quad \mathbb{R}^N,$$

where

$$f_\lambda = -\breve{u}_\lambda + \hat{u}_\lambda + u_\lambda(\delta)\, \psi_1(\delta) - u_\lambda(T)\, \psi_1(T) - d \cdot \nabla(\widehat{|u_\lambda|^{1/N} u_\lambda}) \equiv \sum_{i=1}^{5} J_i.$$

Repeating the argument of the step 4 of the proof of Theorem 2 we pass to the limit in J_i for $i = 1, ..., 4$.

Finally, to pass to the limit in the non-linear term we use the dominated convergence Theorem. By (91)–(92) we have

$$u_{\lambda_n}(t) \to u^*(t) \qquad \text{in} \quad L^p(\Omega) \qquad \text{as} \quad \lambda_n \to \infty$$

for every $t \in [t_1, t_2]$, $1 \leqslant p < \infty$ and every bounded set Ω of $\mathbb{R}^N$. Thus, we deduce that

$$|u_{\lambda_n}|^{1/N} u_{\lambda_n}(t) \to |u^*|^{1/N} u^*(t) \qquad \text{in} \quad L^2(\Omega) \qquad \text{as} \quad \lambda_n \to \infty$$

for every $t \in [t_1, t_2]$. Let $\varphi \in W^{1,1}(t_1, t_2)$ be with support in $[\tau, T]$ with $t_1 < \tau < T < t_2$. Then

$$|u_{\lambda_n}|^{1/N} u_{\lambda_n}(t)\, \varphi(t) \to |u^*|^{1/N} u^*(t)\, \varphi(t) \qquad \text{in} \quad L^2(\Omega) \qquad \text{as} \quad \lambda_n \to \infty$$

for every $t \in (\tau, T)$.

Let $g(t)=ct^{(-N-2)/4}\mathscr{X}_{\text{supp}\,\varphi}\in L^1(0,\infty)$ (where $\mathscr{X}$ denotes the characteristic function). Using (69) we obtain

$$\| |u_{\lambda_n}|^{1/N+1}\,(t)\,\varphi(t)\|_{L^2(\Omega)}\leqslant Cg(t)\in L^1(0,\infty).$$

By the dominated convergence Theorem we have

$$\int_\tau^T |u_{\lambda_n}|^{1/N}\,u_{\lambda_n}\varphi\to\int_\tau^T |u^*|^{1/N}\,u^*\varphi \qquad \text{in} \quad L^2(\Omega) \tag{93}$$

as $\lambda_n\to\infty$ for every $\varphi\in W^{1,\infty}(t_1,t_2)$. Therefore

$$\int_\Omega (\widehat{|u_{\lambda_n}|^{1/N}}\,u_{\lambda_n})\cdot\nabla\psi_2\,dx\to\int_\Omega (\widehat{|u^*|^{1/N}}\,u^*)\cdot\nabla\psi_2\,dx \qquad \text{as} \quad \lambda_n\to\infty$$

for every $\psi_2\in H^1(\mathbb{R}^N)$, i.e.,

$$-\nabla(\widehat{|u_{\lambda_n}|^{1/N}}\,u_{\lambda_n})\to-\nabla(\widehat{|u^*|^{1/N}}\,u^*) \qquad \text{weakly in} \quad H^{-1}(\,\Omega). \tag{94}$$

On the other hand, thanks to (69) and (71) we have

$$\begin{aligned}\|\nabla(\widehat{|u_{\lambda_n}|^{1/N}}\,u_{\lambda_n})\|_2^2&\leqslant C\int_\delta^T \|\nabla(|u_{\lambda_n}|^{1N}\,u_{\lambda_n})\|_2^2\\ &\leqslant C\int_\delta^T\int_{\mathbb{R}^N}|u_{\lambda_n}|^{2/N}\,|\nabla u_{\lambda_n}|^2\,dx\,dt\\ &\leqslant C\int_\delta^T\int_{\mathbb{R}^N}|u_{\lambda_n}|^{2/N}\,|\nabla u_{\lambda_n}|^2\,dx\,dt\leqslant C.\end{aligned}$$

Therefore taking into account (94) and that $\{\nabla(\widehat{|u_{\lambda_n}|^{1/N}}\,u_{\lambda_n})\}$ is uniformly bounded in $L^2(\Omega)$ we deduce that

$$-\nabla(\widehat{|u_{\lambda_n}|^{1/N}}\,u_{\lambda_n})\to-\nabla(\widehat{|u^*|^{1/N}}\,u^*) \text{ weakly } L^2(\Omega).$$

As $L^2(\Omega)$ is compactly embedded in $H^{-1}(\Omega)$, we can conclude that

$$-\nabla(\widehat{|u_{\lambda_n}|^{1/N}}\,u_{\lambda_n})\to-\nabla(\widehat{|u^*|^{1/N}}\,u^*) \qquad \text{strongly} \quad H^{-1}(\,\Omega).$$

Therefore we have deduced that u^* satisfies the following equality:

$$\int_{\mathbb{R}^N} [a^h \nabla \hat{u}^* \cdot \nabla \psi_2 + \check{u}^* \psi_2] \, dx$$

$$= -\int_{\mathbb{R}^N} u^*(T) \, \psi_2 \psi_1(T) \, dx + \int_{\mathbb{R}^N} u^*(\delta) \, \psi_2 \psi_1(\delta) \, dx$$

$$+ \int_{\mathbb{R}^N} d \cdot \nabla \psi_2 \, \widehat{|u_\lambda|^{1/N}} \, u_\lambda \, dx. \tag{95}$$

Finally, we claim that

$$\lim_{\delta \to 0^+} \int_{\mathbb{R}^N} u^*(\delta, x) \, \psi_2(x) \, dx = M\psi_2(0) \tag{96}$$

for every $\psi_2 \in BC^2(\mathbb{R}^N)$.

Multiplying equation (9) by a test function $\psi_2 \in C_c^\infty(\mathbb{R}^N)$ and integrating in $(0, \delta) \times \mathbb{R}^N$ we have:

$$\left| \int_{\mathbb{R}^N} u_\lambda(\delta, x) \, \psi_2(x) - \int_{\mathbb{R}^N} u_\lambda(0, x) \, \psi_2(x) \right|$$

$$\leqslant \left| \int_0^\delta \int_{\mathbb{R}^N} a(\lambda x) \, \nabla u_\lambda(s, x) \cdot \nabla \psi_2(x) \, dx \, ds \right|$$

$$+ \left| d \cdot \int_0^t \int_{\mathbb{R}^N} |u_\lambda(s, x)|^{q-1} \, u_\lambda(s, x) \, \nabla \phi(x) \, dx \, ds \right| \equiv |I_1| + |I_2|.$$

Repeating the same arguments used in previous sections we deduce that for any $\varepsilon > 0$ there exist some $\tau > 0$ and $\lambda_0 > 1$ such that $|I_1| < \varepsilon$ if $0 < \delta < \tau$ and $\lambda > \lambda_0$.

Now, we consider $|I_2|$. In this case we have

$$\left| \int_0^\delta \int_{\mathbb{R}^N} d \cdot \nabla \psi_2(x) \, |u_\lambda|^{1/N} \, u_\lambda \, dx \, dt \right|$$

$$\leqslant C \, |d| \int_0^\delta \| |u_\lambda|^{1+1/N} (s) \|_1 \, ds \leqslant C \, |d| \, \|u_0\|_1^{1+1/N} \, \delta^{1/2}$$

that tends to zero when $\delta \to 0$ uniformly for $\lambda \geqslant \lambda_0 > 0$. Now, using the arguments of previous sections we deduce (96) for every bounded and continuous function ψ_2.

Therefore u^* is a weak solution of

$$\begin{cases} u^*_t - \operatorname{div}(a^h \nabla u^*) = d \cdot \nabla(|u^*|^{1/N} \, u^*) & \text{in} \quad (0, \infty) \times \mathbb{R}^N \\ u^*(0, x) = M\delta. \end{cases} \tag{97}$$

Step 3: *Uniqueness of the solution of the limit problem.* In the above step we have proved that there exists a subsequence $\{u_{\lambda_n}\}$ such that

$$u_{\lambda_n}(t) \to u^*(t) \qquad \text{strongly in} \quad L^p_{loc}(\mathbb{R}^N)$$

as $\lambda_n \to \infty$, for every $1 \leqslant p < \infty$ and for every $t > 0$ and that $u^*(x, t)$ satisfies the equation (97).

Since we are assuming the convection direction d to be costant, we may assume without loss of generality that $d = -e_N = (0, ..., -1)$. We denote by (x, y) a generic point of $\mathbb{R}^N$ with $x = (x_1, ..., x_{N-1}) \in \mathbb{R}^{N-1}$ and $y \in \mathbb{R}$. With this notation (97) becomes

$$u^*_t - \operatorname{div}(a^h \nabla u^*) - \partial_y(|u^*|^{1/N} u^*) = 0. \tag{98}$$

The uniqueness theorem of [4] guarantees that the solution u^* of (97) is unique provided it satisfies

$$\lim_{t \to 0^+} \int_{\mathbb{R}^N} u(t, x, y)\, \phi(x, y)\, dx = M\phi(0, 0) \tag{99}$$

for every $\phi \in BC(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^{N-1}} dx \int_{|y| > R} dy\, |u|(t, x, y) \to 0 \qquad \text{as} \quad t \to 0 \tag{100}$$

for some $R > 0$ fixed.

We multiply equation (90) by a test function $\phi \in C^\infty_c(\mathbb{R}^N)$. Integrating in $(0, t) \times \mathbb{R}^N$ we have:

$$\left| \int_{\mathbb{R}^N} u_\lambda(t, x)\, \phi(x)\, dx - \int_{\mathbb{R}^N} u_\lambda(0, x)\, \phi(x)\, dx \right|$$
$$\leqslant \left| \int_0^t \int_{\mathbb{R}^N} [u_\lambda a_{ij}(\lambda x)\, \partial^2_{ij}\phi + u_\lambda \lambda\, \partial_i a_{ij}(\lambda x)\, \partial_j \phi\, dx\, ds \right]$$
$$+ - \int_0^t \int_{\mathbb{R}^N} (|u_\lambda)|^{1/N} u_\lambda)\, \nabla\phi(x)\, dx\, ds \Bigg| \equiv |I_1|.$$

In step 2 we have proved that for any $\varepsilon > 0$ there exists some $\tau > 0$ and $\lambda_0 > 1$ such that $|I_1| \leqslant \varepsilon$, for every $0 < t \leqslant \tau$ and $\lambda \geqslant \lambda_0$. Therefore taking this into account and using that

$$\int_{\mathbb{R}^N} u_\lambda(0, x)\, \phi(x)\, dx \to M\phi(0) \qquad \text{as} \quad \lambda \to \infty$$

we conclude that for any $\varepsilon > 0$ there exist $\tau > 0$ and $\lambda_0 > 1$ such that

$$\left| \int_{\mathbb{R}^N} u_\lambda(t, x)\, \phi(x)\, dx - M\phi(0) \right| < \varepsilon$$

if $0 < t < \tau$ and $\lambda > \lambda_0$. It then follows that (99) holds for every $\phi \in C_c^\infty(\mathbb{R}^N)$. On the other hand, taking (75) into account we deduce (99) for every bounded and continuous function ϕ.

Now, let us prove (100).

Applying Fatou's Lemma we have

$$\int_{\mathbb{R}^{N-1}} dx \int_{|y| > R} dy\, |u|(t, x, y) \leqslant \lim_{\lambda_n \to \infty} \int_{\mathbb{R}^{N-1}} dx \int_{|y| > R} dy\, |u_{\lambda_n}|(t, x, y).$$

To prove (100) it is enough to prove that for some $R > 0$ fixed

$$\int_{\mathbb{R}^{N-1}} dx \int_{|y| > R} dy\, |u_{\lambda_n}|(t, x, y) \to 0$$

by letting first $\lambda_n \to \infty$ and then $t \to 0$.

By the comparison principle of solutions (see Lemma 1 of [9]) we have

$$|u_{\lambda_n}| \leqslant \bar{u}_{\lambda_n} \quad \forall t > 0, \qquad \text{a.e.} \quad x \in \mathbb{R}^N,$$

where $\bar{u}_{\lambda_n}$ is the solution of (97) with initial data $|u_{\lambda_n}|(0)$. Therefore we deduce that

$$\int_{\mathbb{R}^{N-1}} dx \int_{|y| > R} dy\, |u_{\lambda_n}|(t, x, y) \leqslant \int_{\mathbb{R}^{N-1}} dx \int_{|y| > R} dy\, \bar{u}_{\lambda_n}(t, x, y).$$

Now, we introduce the function $\varphi_R(y) = \varphi(y/R)$ with $\varphi \in BC^1(\mathbb{R})$ such that

$$\varphi(y) = \begin{cases} 0, & |y| < \frac{1}{2} \\ 1, & |y| > 1 \end{cases}$$

and $0 \leqslant \varphi \leqslant 1$ if $1/2 < |y| < 1$.

Multiplying the equation satisfied by $\bar{u}_{\lambda_n}$ by φ_R and integrating we have

$$\frac{d}{dt} \int \bar{u}_{\lambda_n}(t, x, y)\, \varphi_R(y)\, dy\, dx = \int \bar{u}_{\lambda_n}(t, x, y)\; \frac{1}{R} \lambda_n\, \partial_i a_{iy}(\lambda_n x) (\partial_y \varphi)_R\, (x)\, dy\, dx$$

$$+ \int (\bar{u}_{\lambda_n})^{1+1/N} \frac{1}{R} (\partial_y \varphi)_R\, (x)\, dy\, dx,$$

where $(\partial_y \varphi)_R(x) = \partial_y \varphi(\frac{y}{R})$. Thanks to estimates (69) and (70) and that $b_{ij} \in L^\infty(\mathbb{R}^N)$ we have

$$\frac{d}{dt}\int \bar{u}_\lambda(t, x)\, \varphi_R(x)\, dx$$

$$\leqslant \frac{1}{R}\left[\|\lambda_n\, \partial_i a_{iy}(\lambda_n x)\|_p\, \|\partial_y \varphi\|_\infty\, \|\bar{u}_{\lambda_n}\|_{p'} + C\, \|\partial_y \varphi\|_\infty\, \|\bar{u}_{\lambda_n}\|_\infty^{1/N}\right]$$

$$\leqslant \frac{C}{R}\lambda_n^{1-N/p}\, t^{(-N/2)(1-1/p')} + \frac{C}{R}\, t^{-1/2}$$

taking $\frac{N}{2} < p \leqslant N$. Therefore we obtain that

$$\int \bar{u}_{\lambda_n}(t, x, y)\, \varphi_R(x)\, dy\, dx \leqslant \int \bar{u}_{\lambda_n}(0, x, y)\, \varphi_R(y)\, dy\, dx + \frac{C}{R}\, t^{1/2} + \frac{C}{R}\, t^\varepsilon \lambda_n^{1-N/p}$$

with $\varepsilon > 0$.

As we have

$$\int_{\mathbb{R}^{N-1}} dx \int_{|y|>R} dy\, \bar{u}_{\lambda_n}(t, x, y) \leqslant \int \bar{u}_{\lambda_n}(t, x, y)\, \varphi_R(y)\, dy\, dx$$

we obtain that

$$\int_{\mathbb{R}^{N-1}} dx \int_{|y|>R} dy\, |u_{\lambda_n}|(t, x, y) \leqslant \int \bar{u}_{\lambda_n}(0, x, y)\, \varphi_R(y)\, dy\, dx + \frac{C}{R}\, t^{1/2}.$$

Thus, it is sufficient to prove that

$$\int \bar{u}_{\lambda_n}(0, x, y)\, \varphi_R(y)\, dy\, dx + \frac{C}{R}\, t^{1/2} + \frac{C}{R}\, t^\varepsilon \lambda_n^{1-N/p} \to 0$$

as $\lambda_n \to \infty$ and $t \to 0$ for some $R > 0$ fixed.

Taking into account that

$$\int \bar{u}_{\lambda_n}(0, x, y)\, \varphi_R(y)\, dy\, dx \leqslant \int_{\mathbb{R}^{N-1}} dx \int_{|y|>R/2} dy\, {\lambda_n}^N \bar{u}(0, \lambda_n x, \lambda_n y)\, dx$$

and rescaling we obtain

$$\int_{\mathbb{R}^{N-1}} dx \int_{|y|>R/2} dy\, {\lambda_n}^N \bar{u}(0, \lambda_n x, \lambda_n y)\, dx = \int_{\mathbb{R}^{N-1}} dz \int_{|z_n|>\lambda_n R/2} \bar{u}(0, z, z_n)\, dz_n.$$

Since $\bar{u}(0) \in L^1(\mathbb{R}^N)$ we deduce that

$$\int_{\mathbb{R}^{N-1}} dz \int_{|z_n| > \lambda_n R/2} \bar{u}(0, z, z_n)\, dz_n + \frac{C}{R} t^{1/2} \to 0$$

as $\lambda_n \to \infty$ and $t \to 0$ for any $R > 0$ fixed. Therefore, according to [4], we conclude that the solution u of (97) is unique.

The uniqueness of the limit and (91) forces the whole family $\{u_\lambda\}$ to strongly converge to u in $C([t_1, t_2]; H^{-\varepsilon}(\Omega))$ as $\lambda \to \infty$.

The rest of the proof of Theorem 3 is concluded as the proof of Theorem 1. ∎

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